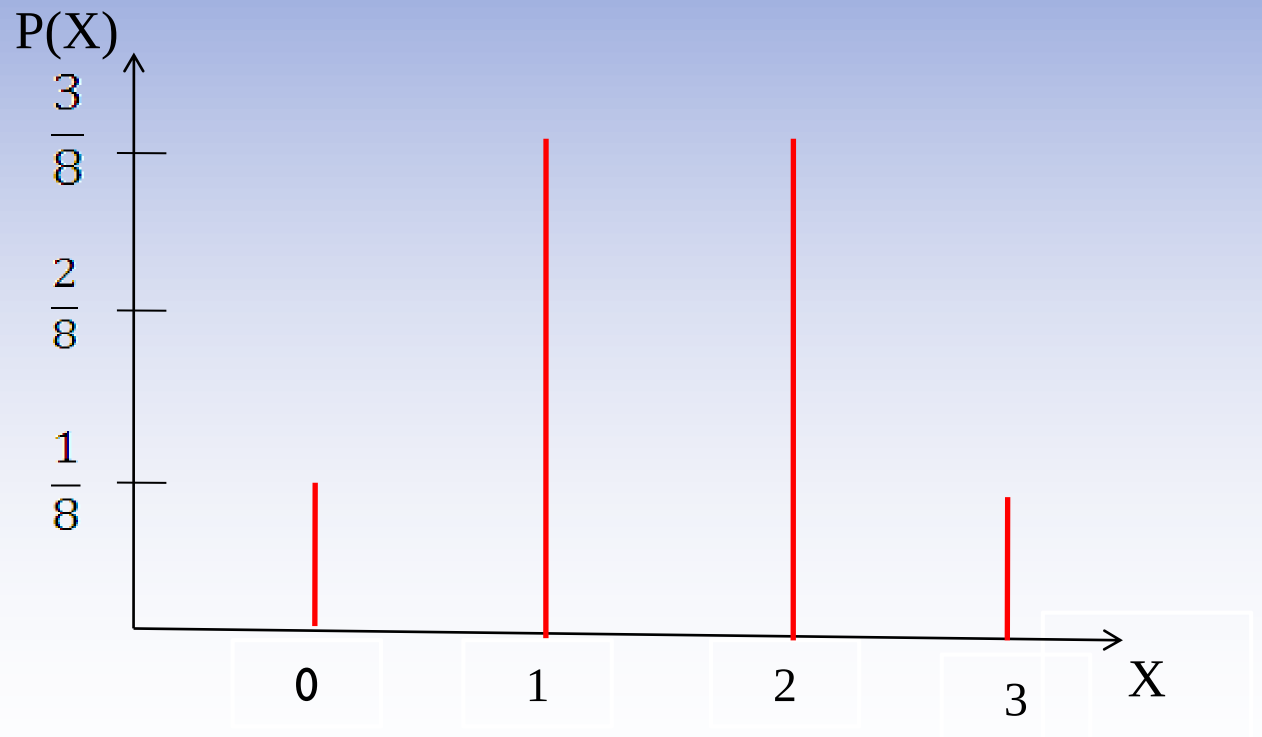
Continuous Random Variables and Probability distribution

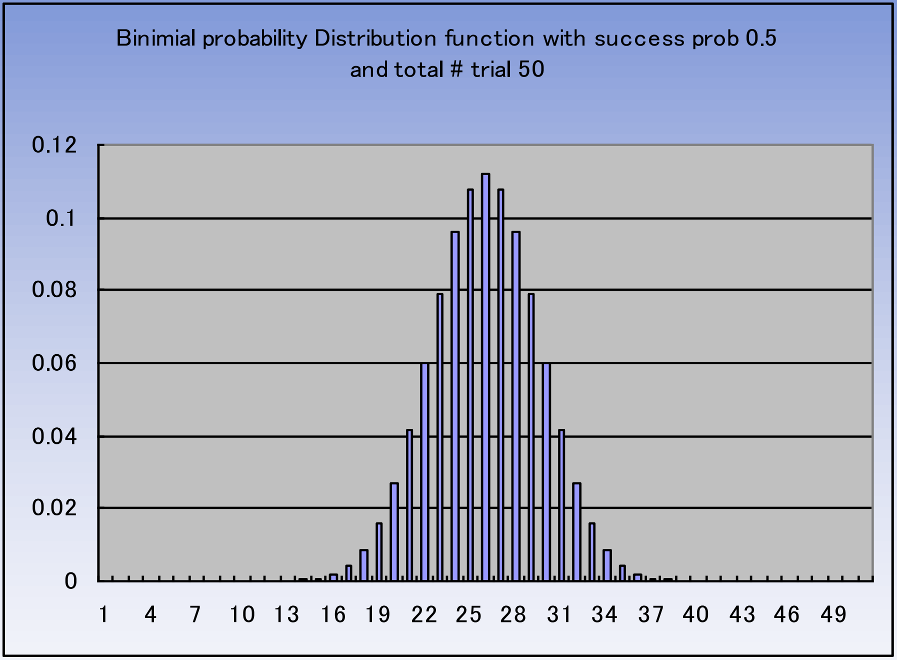
Tossing Coins: Discrete Random Variable

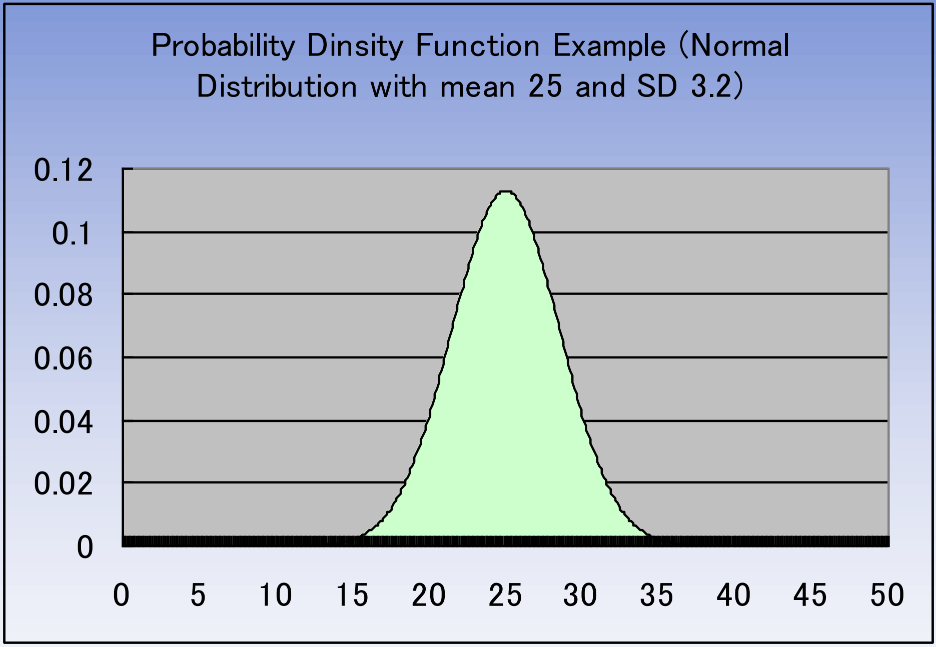
Represent graphically the probability distribution for the sample space for tossing three coins .

X	0	1	2	3
P(X)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

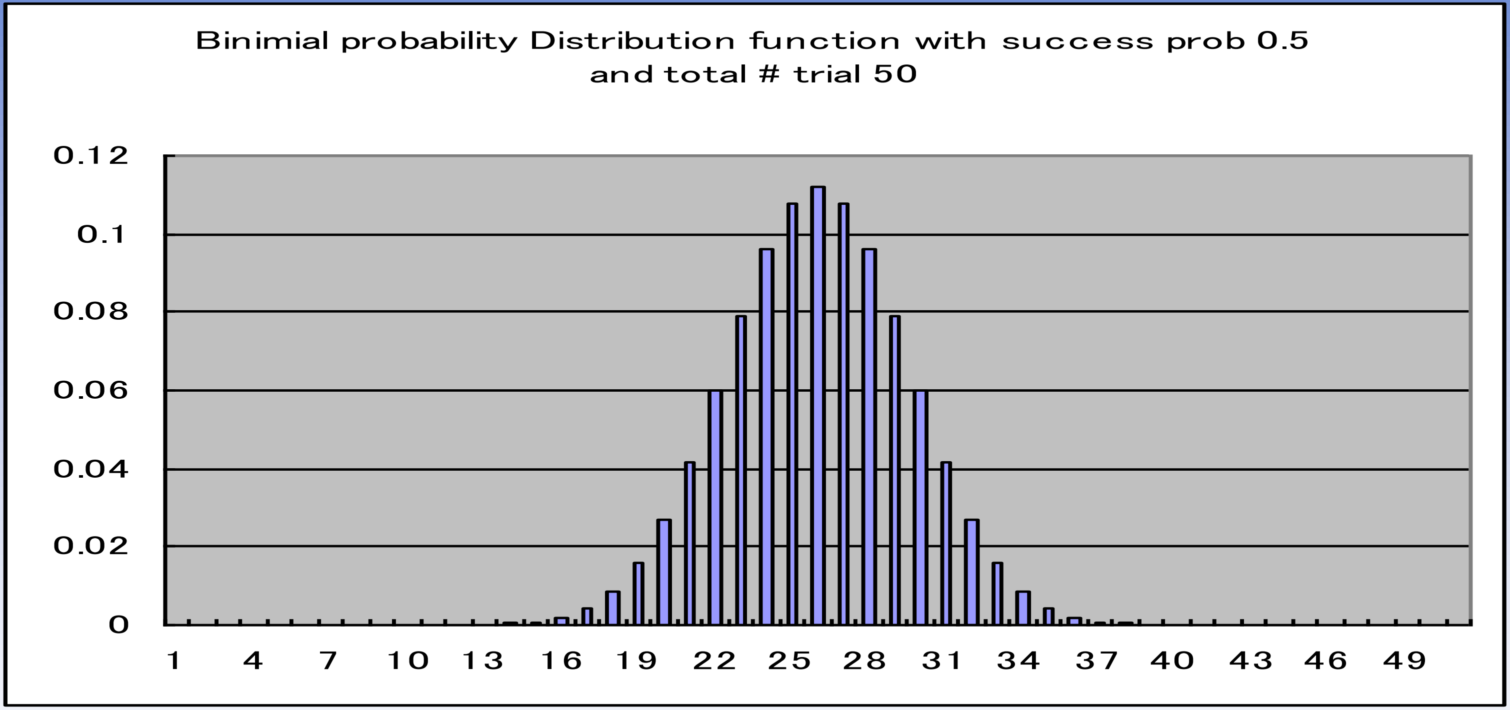


Comparing Probability Distribution Function (for discrete r.v) and Probability density function (for continuous r.v)



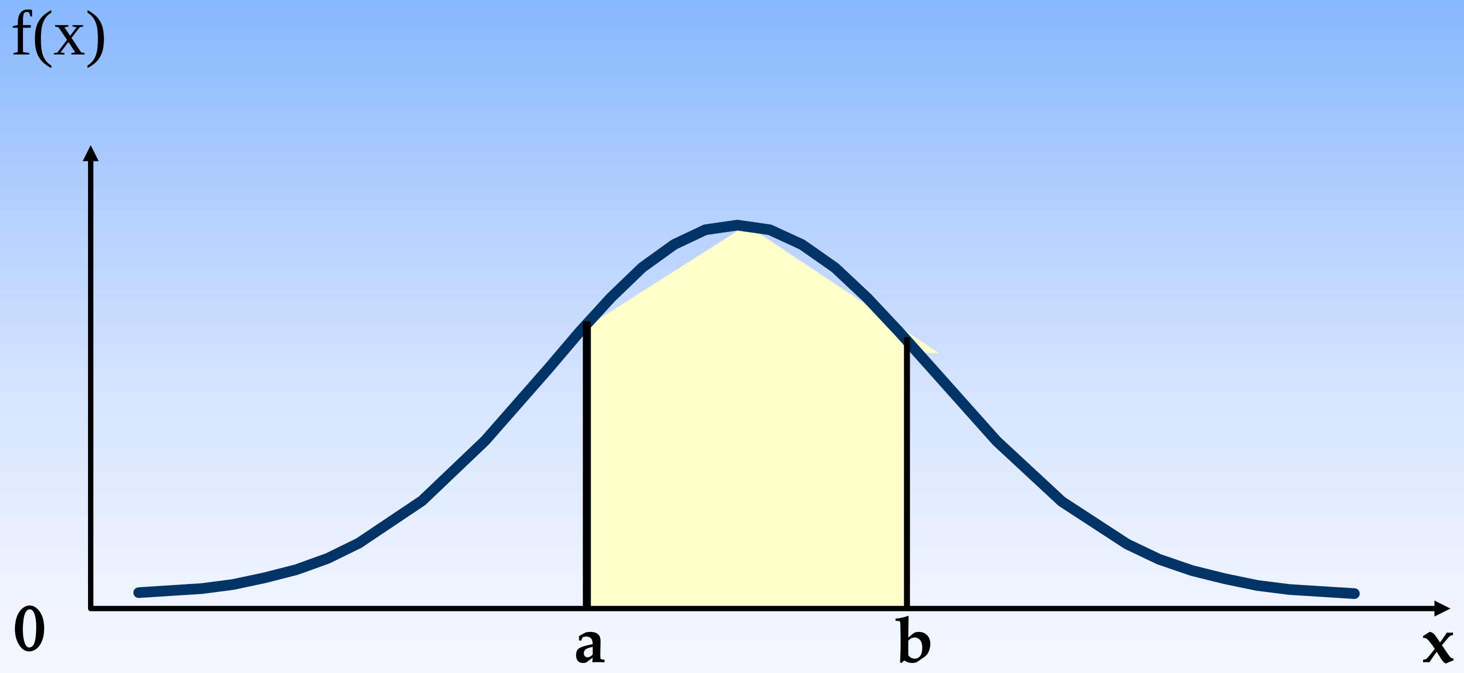


Comparing Probability Distribution Function (Discrete r.v) and Probability density function



Probability distribution function for a discrete random variable shows the probability for each outcome.

Comparing Probability Distribution Function (Discrete r.v) and Probability density function (Continuous r.v)



Probability Density Function shows the probability that the random variable falls in a particular range.

Continuous Random Variables

A random variable is **continuous** if it can take any value in an interval.

Probability Distribution of Continuous Random Variables

- For continuous random variable, we use the following two concepts to describe the probability distribution

1. Probability Density Function
2. Cumulative Distribution Function

Probability Density Function

- Probability Density Function is a similar concept as the probability distribution function for a discrete random variable.
- You can consider the probability density function as a "smoothed" probability distribution function.

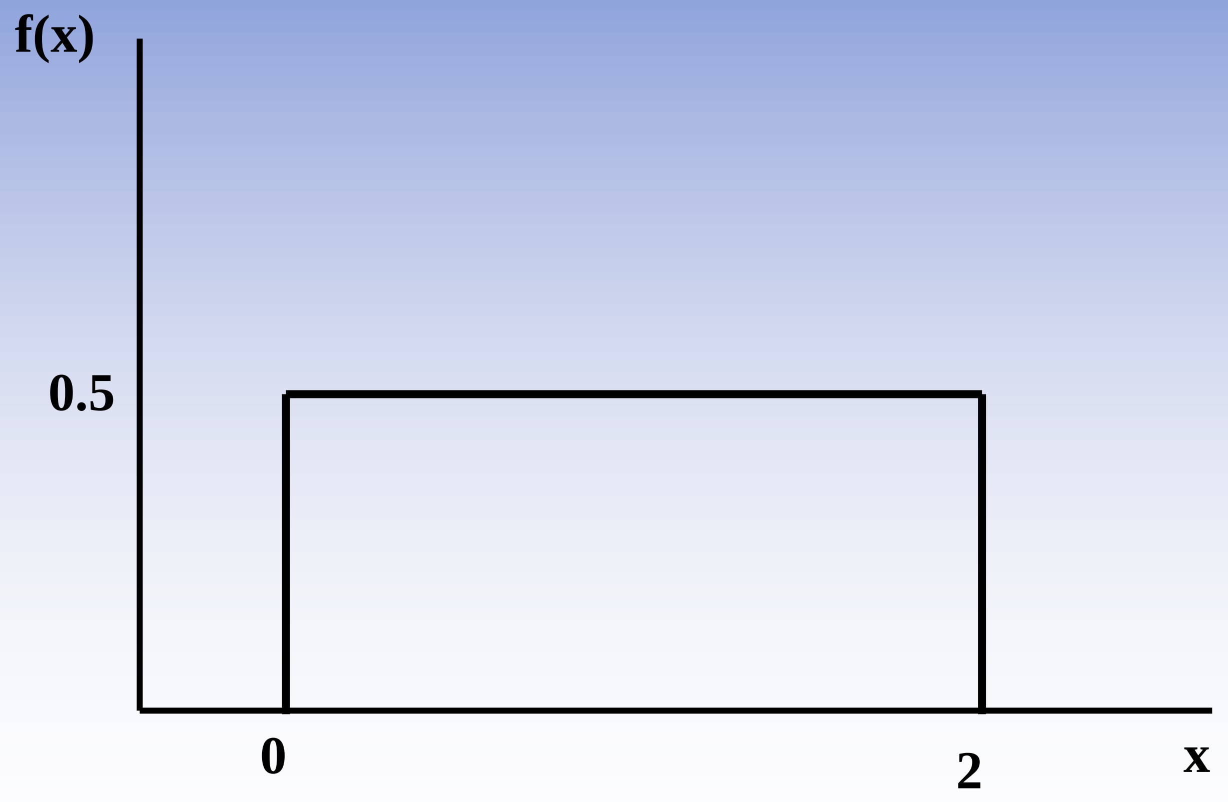
Probability density function

- Let X be a continuous random variable. Let x denotes the value that the random variable X takes. We use f(x) to denote the probability density function.

Example

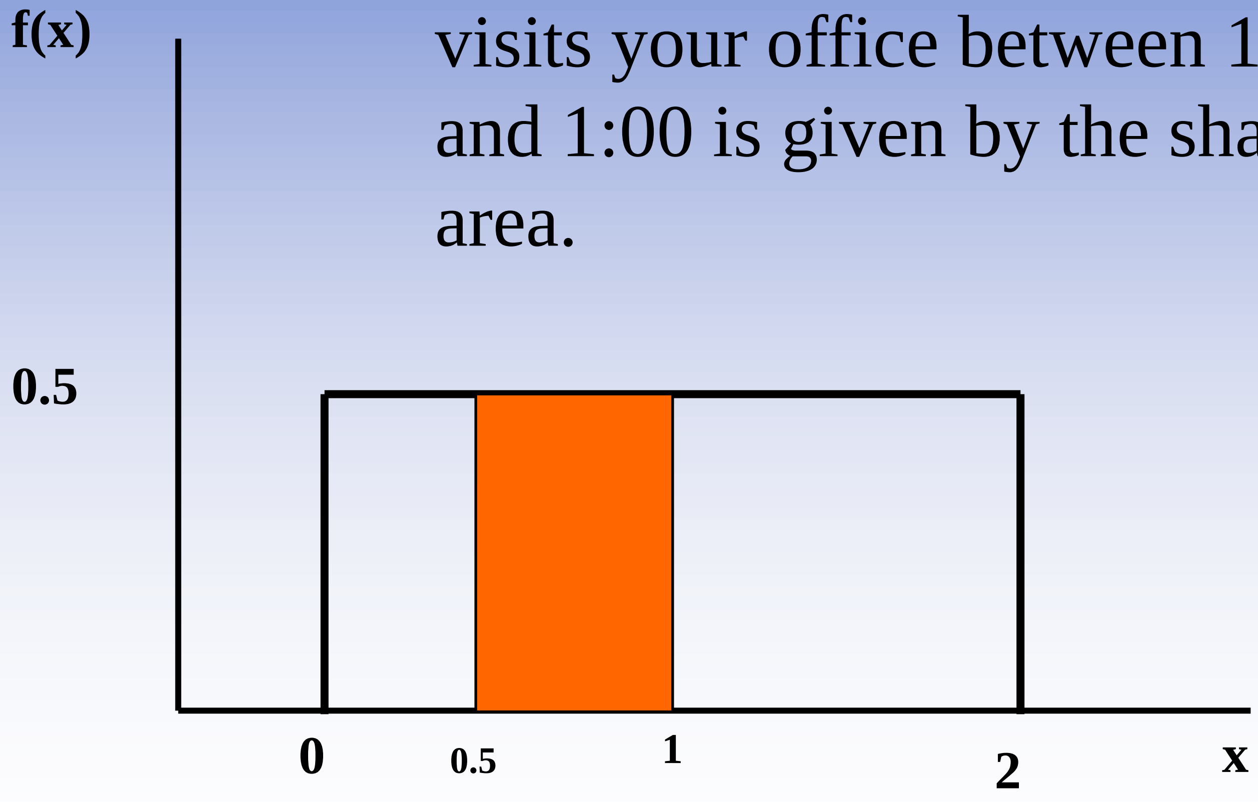
- You client told you that he will visit you between noon and 2pm. Between noon and 2pm, the time he will arrive at your company is totally random. Let X be the random variable for the time he arrives (X=1.5 means he visit your office at 1:30pm)
- Let x be the possible value for the random variable X. Then, the probability density function f(x) has the following shape.

Probability Density Function Example (Uniform Probability Distribution)



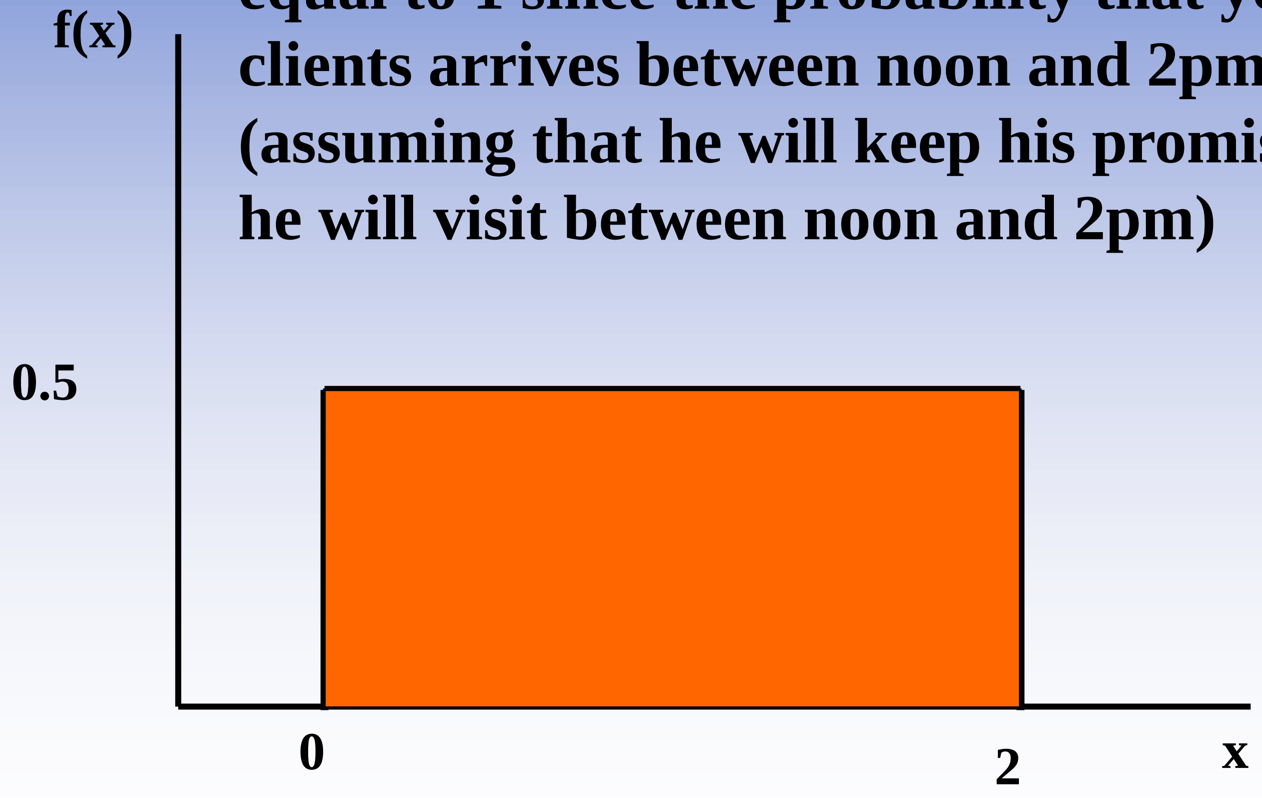
Probability Density Function Example

The probability that your client visits your office between 12:30 and 1:00 is given by the shaded area.



Probability Density Function Example

Note that area between 0 and 2 should be equal to 1 since the probability that your clients arrives between noon and 2pm is 1 (assuming that he will keep his promise that he will visit between noon and 2pm)



Some Properties of probability density function

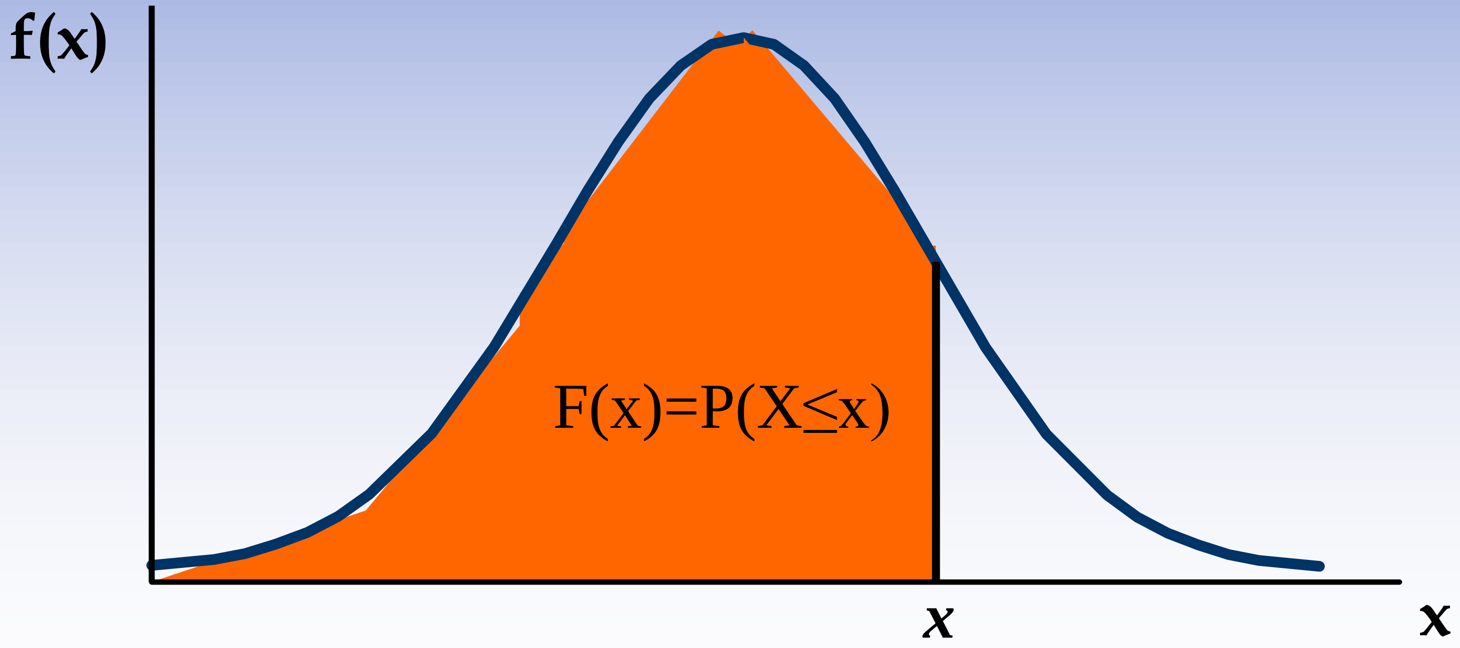
1. $f(x) \geq 0$ for any x
2. Total area under f(x) is 1

Cumulative Distribution Function

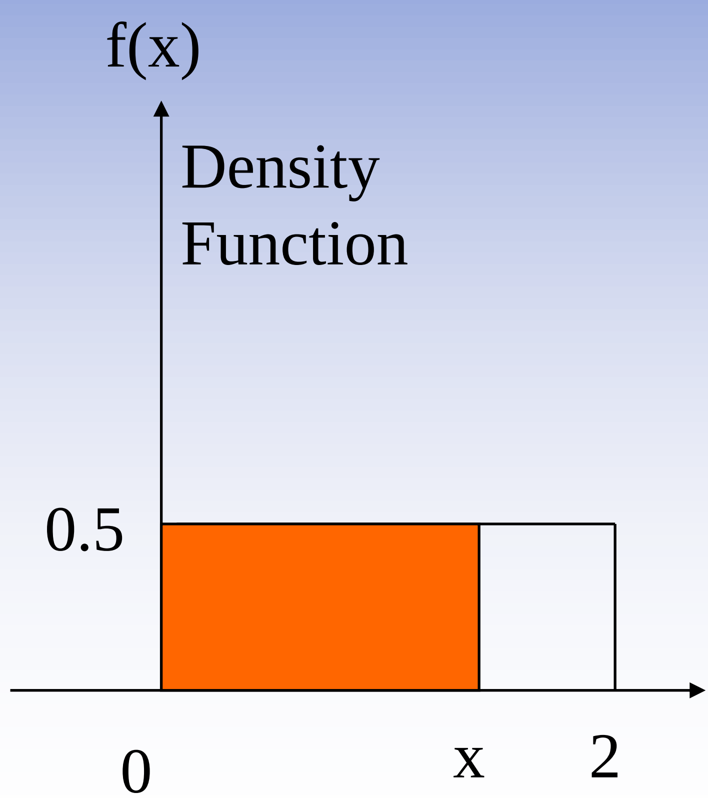
The **cumulative distribution function**, F(x), for a continuous random variable X expresses the probability that X does not exceed the value of x, as a function of x

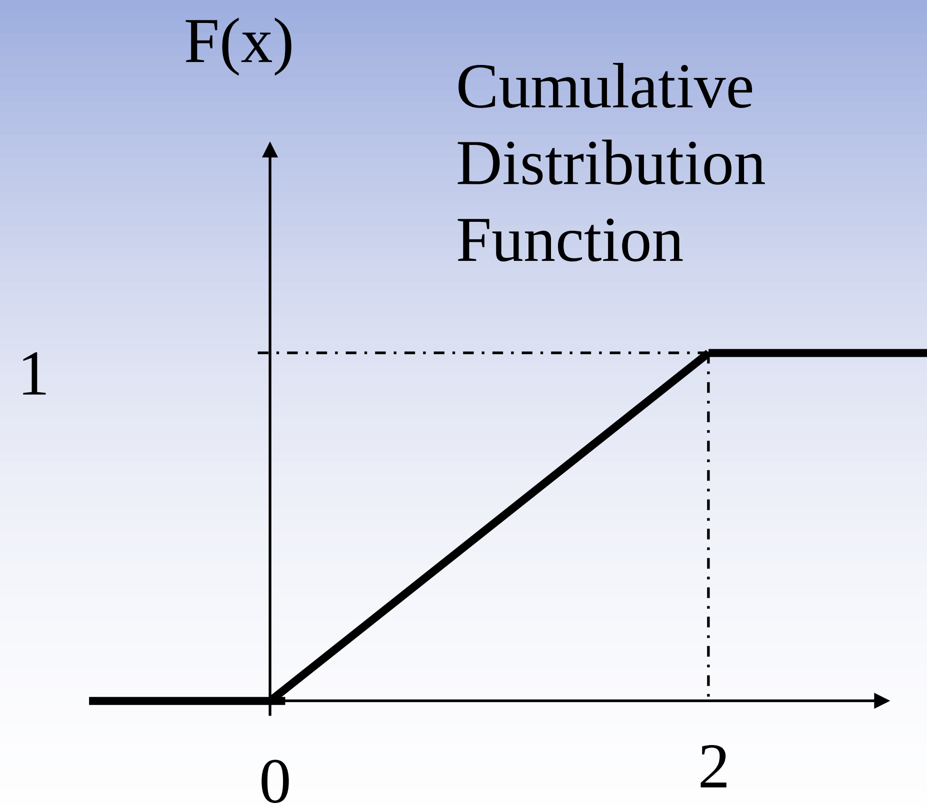
$$F(x) = P(X \leq x)$$

In other words, the cumulative distribution function F(x) is given by the shaded area.



Cumulative Distribution Function
-Example-





A property of cumulative distribution functions

$$P(a < X < b) = P(X < b) - P(X < a)$$

$$= F(b) - F(a)$$

Relationship Between Probability Density Function and Cumulative Distribution Function

- Let X be a continuous random variable. Then, there is a following relationship between probability density function and cumulative distribution function.

$$P(a < X < b) = F(b) - F(a)$$

$$= \int_a^b f(u)du$$

Continuous Random Variables

$f(x) \geq 0$ for any x

Total area under f(x) is 1

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$F(x) = P(X \leq x)$$

$$P(a < X < b) = P(X < b) - P(X < a)$$

$$= F(b) - F(a) = \int_a^b f(x)dx$$

Example (1)

$$f(x) = \begin{cases} \frac{x^2}{2}, & -1 \leq x \leq 2 \\ 0, & otherwise \end{cases}$$

a) Verify that $f(x)$ is a Probability density function.

b) Find $P(0 \leq x \leq 1)$

Solution

a) Verify that *f(x)* is a Probability density function.

1. We have to prove that $f(x) \geq 0$ for every value in the interval.

2. $\int_{-\infty}^{\infty} f(x)dx = 1$

In our example we have to prove that

$$\int_{-1}^{2} f(x)dx \; must\ equal\ to\ 1$$

Or $$\int_{-1}^{3} \frac{x^2}{3} dx = 1$$

$$\int_{-1}^{2} \frac{x^2}{3} dx = \left[\frac{x^3}{9}\right]_{-1}^{2} = \frac{2^3}{9} - \frac{(-1)^3}{9} = \frac{8}{9} + \frac{1}{9} = \frac{9}{9} = \mathbf{1}$$

So, f(x) is a probability density function

Solution

b) Find $P(0 \leq x \leq 1)$

$$P(0 \leq x \leq 1) = \int_0^1 \frac{x^2}{3}\,dx$$

$$= \left[\frac{x^3}{9}\right]_0^1 = \frac{1^3}{9} - \frac{(0)^3}{9} = \frac{1}{9} - 0 = \frac{1}{9}$$

Example (2)

$$f(x) = \begin{cases} \frac{2x+1}{24}, 2 \le x \le 5 \\ 0, otherwise \end{cases}$$

a) Verify that $f(x)$ is a Probability density function.
b) Find $P(2.5 \le x \le 4.5)$

Solution

a) Verify that *f(x)* is a Probability density function.

1. We have to prove that $f(x) \geq 0$ for every value in the interval.

x	2	3	4	5
$f(x)$	0.208333	0.291667	0.375	0.458333

2. $\int_{-\infty}^{\infty} f(x)dx = 1$, in our example we have to prove that

$$\int_{2}^{5} f(x)dx \; must\ equal\ to\ 1$$

Or $\int_{2}^{5} \frac{2x+1}{24} dx = 1$

$$\int_{2}^{5} \frac{2x+1}{24} dx = \left[\frac{x^2+x}{24}\right]_2^5 = \frac{5^2+5}{24} - \frac{2^2+2}{24} = \frac{30}{24} - \frac{6}{24} = \frac{24}{24} = \mathbf{1}$$

So, f(x) is a probability density function

Solution

b) Find $P(2.5 \leq x \leq 4.5)$

$$P(2.5 \leq x \leq 4.5) = \int_{2.5}^{4.5} \frac{2x+1}{24} dx$$

$$= \int_{2.5}^{4.5} \frac{2x+1}{24} dx = \left[\frac{x^2+x}{24}\right]_{2.5}^{4.5} = \frac{(4.5)^2+4.5}{24} - \frac{(2.5)^2+2.5}{24} =$$

$$\frac{33}{32} - \frac{35}{96} = \frac{2}{3}$$

Example (3)

$$f(x) = \begin{cases} \frac{x+3}{18}, & -3 \le x \le 3 \\ 0, & otherwise \end{cases}$$

a) Verify that *f(x)* is a Probability density function.
b) Find $P(x \le 0)$
c) Find $P(-1 \le x \le 4)$

Solution

a) Verify that *f(x)* is a Probability density function.

1. We have to prove that $f(x) \geq 0$ for every value in the interval.

x	-3	-2	-1	0	1	2	3
$f(x)$	0	0.055556	0.111111	0.166667	0.222222	0.277778	0.333333

2. $\int_{-\infty}^{\infty} f(x)dx = 1$, in our example we have to prove that

$$\int_{-3}^{3} f(x)dx \; must \; equal \; to \; 1$$

Or $$\int_{-3}^{3} \frac{x+3}{18} dx = 1$$

So, f(x) is a probability density function

Solution

$$f(x) = \begin{cases} \frac{x+3}{18}, & -3 \le x \le 3 \\ 0, & otherwise \end{cases}$$

b) Find $P(x \le 0)$

$$P(x \le 0) = P(-3 \le x \le 0) + P(x < -3)$$

$$\int_{-3}^{0} \frac{x+3}{18}\,dx + 0 = \left[\frac{\frac{1}{2}x^2 + 3x}{18}\right]_{-3}^{0} + 0 = \frac{1}{4} + 0 = \frac{1}{4}$$

Solution

c) Find P(-1 ≤ x ≤ 4)

$$f(x) = \begin{cases} \frac{x+3}{18}, & -3 \le x \le 3 \\ 0, & otherwise \end{cases}$$

$$P(-1 \le x \le 4) = P(-1 \le x \le 3) + P(3 < x \le 4)$$

$$\int_{-1}^{3} \frac{x+3}{18}\,dx + 0 = \left[\frac{\frac{1}{2}x^2+3x}{18}\right]_{-1}^{3} + 0 = \frac{8}{9} + 0 = \frac{8}{9}$$

Example (4)

Given that f(x) is a probability density function

$$f(x) = \begin{cases} \frac{2x+2}{a}, 2 \le x \le 5 \\ 0, \quad otherwise \end{cases}$$

a) Find the value of *a*
b) Find $P(-1 \le x \le 3)$

Solution

a) Find the value of a

$$P(2 \leq x \leq 5) = \int_2^5 \frac{2x+2}{a}\,dx = 1$$

$$\left[\frac{x^2+2x}{a}\right]_2^5 = \frac{5^2+2(5)}{a} - \frac{2^2+2(2)}{a} = \frac{35}{a} - \frac{8}{a} = \frac{27}{a} = 1$$

$$a = 27$$

So the probability density function will be

$$f(x) = \begin{cases} \frac{2x+2}{27}, & 2 \leq x \leq 5 \\ 0, & otherwise \end{cases}$$

Solution

$$f(x) = \begin{cases} \frac{2x+2}{a}, & 2 \le x \le 5 \\ 0, & otherwise \end{cases}$$

b) Find P(-1 ≤ x ≤ 3)

$$P(x \le 0) = P(-1 \le x < 2) + P(2 \le x \le 3)$$

$$0 + \int_{-1}^{3} \frac{2x+2}{27}\, dx = 0 + \left[\frac{x^2+2x}{27}\right]_{-1}^{3} = \frac{7}{27}$$

Example (5)

Given that f(x) is a probability density function

$$f(x) = \begin{cases} \frac{1}{4}, & 0 \le x < 3 \\ \frac{1}{8}(5 - x), & 3 \le x \le 5 \\ 0, & otherwise \end{cases}$$

a) Proof that $f(x)$ is a Probability density function.
b) Find $P(1 \le x \le 4)$

Solution

a) Proof that *f(x)* is a Probability density function.

$$f(x) = \begin{cases} \frac{1}{4}, & 0 \le x < 3 \\ \frac{1}{8}(5-x), & 3 \le x \le 5 \\ 0, & otherwise \end{cases}$$

$\int_{-\infty}^{\infty} f(x)dx = 1$, we have to prove that

$$\int_0^5 f(x)dx \; must\ equal\ to\ 1$$

$$\int_0^5 f(x)dx = \int_0^3 \frac{1}{4}dx + \int_3^5 \frac{1}{8}(5-x) = 1$$

$$\left[\frac{x}{4}\right]_0^3 + \left[\frac{1}{8}\left(5x - \frac{1}{2}x^2\right)\right]_3^5 = \frac{3}{4} + \frac{1}{8}\left[\left(5(5) - \frac{1}{2}(25)\right) - \left(5(3) - \frac{1}{2}(9)\right)\right]$$

$$= \frac{3}{4} + \frac{1}{4} = \frac{4}{4} = 1$$

So, f(x) is a probability density function

Solution

$$f(x) = \begin{cases} \frac{1}{4}, & 0 \le x < 3 \\ \frac{1}{8}(5-x), & 3 \le x \le 5 \\ 0, & otherwise \end{cases}$$

b) Find $P(1 \le x \le 4)$

$$P(1 \le x \le 4) = P(1 \le x < 3) + P(3 \le x \le 4)$$

$$\int_1^3 \frac{1}{4}\,dx + \int_3^4 \frac{1}{8}\,(5-x) =$$

$$\left[\frac{x}{4}\right]_1^3 + \left[\frac{1}{8}\,(5x - \frac{1}{2}\,x^2)\right]_3^4 = (\frac{3}{4} - \frac{1}{4}) + \frac{1}{8}\left[\left(5(4) - \frac{1}{2}\,(16)\right) - \left(5(3) - \frac{1}{2}\,(9)\right)\right]$$

$$\frac{2}{4} + \frac{3}{16} = \frac{11}{16}$$