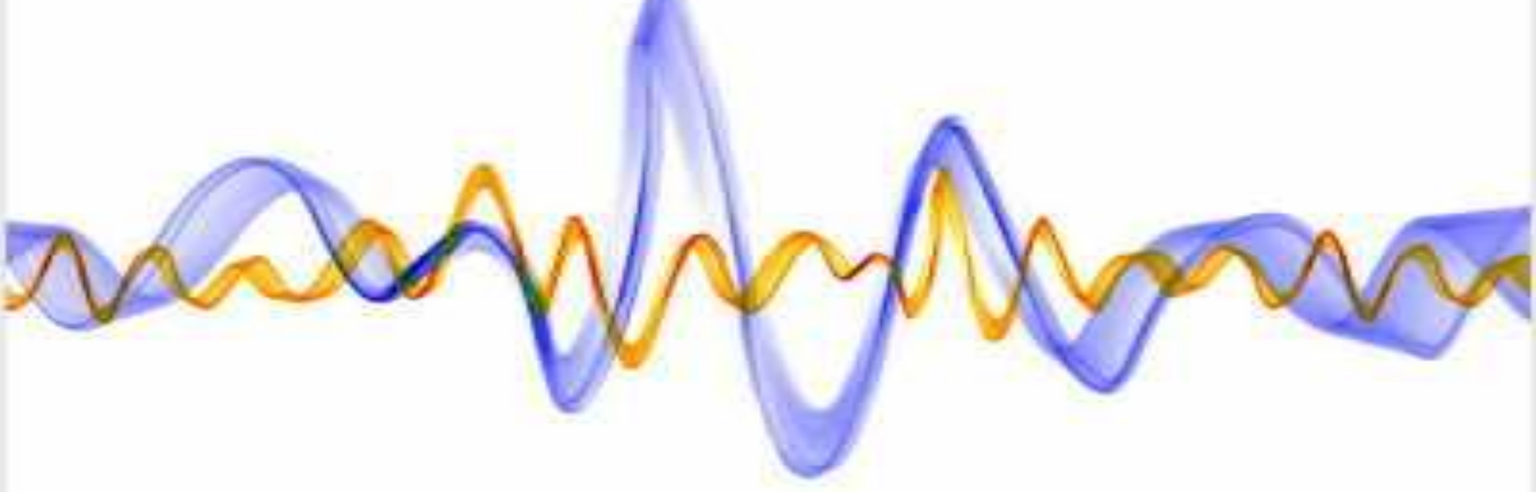
Digital Logic Design



Lecture 3:

Chapter 2: Boolean Algebra and Logic Gates

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Basic Definitions

- **The Postulates Boolean Algebra**

Table 2.1
Postulates and Theorems of Boolean Algebra

Postulate 2	(a)	$x + 0 = x$	(b)	$x \cdot 1 = x$
Postulate 5	(a)	$x + x' = 1$	(b)	$x \cdot x' = 0$
Theorem 1	(a)	$x + x = x$	(b)	$x \cdot x = x$
Theorem 2	(a)	$x + 1 = 1$	(b)	$x \cdot 0 = 0$
Theorem 3, involution		$(x')' = x$		
Postulate 3, commutative	(a)	$x + y = y + x$	(b)	$xy = yx$
Theorem 4, associative	(a)	$x + (y + z) = (x + y) + z$	(b)	$x(yz) = (xy)z$
Postulate 4, distributive	(a)	$x(y + z) = xy + xz$	(b)	$x + yz = (x + y)(x + z)$
Theorem 5, DeMorgan	(a)	$(x + y)' = x'y'$	(b)	$(xy)' = x' + y'$
Theorem 6, absorption	(a)	$x + xy = x$	(b)	$x(x + y) = x$

Basic Definitions

- **Duality Principle (DeMorgan's Theorem)**
- Verify DeMorgan's Theorem

$$(x+y)' = x'y'$$
$$(x\ y)' = x' + y'$$

$$x + y = (x'y')'$$
$$x\ y = (x' + y')'$$

x	y	x'	y'	$x+y$	$(x+y)'$	$x'y'$	Xy	$x'+y'$	$(xy)'$
0	0	1	1	0	1	1	0	1	1
0	1	1	0	1	0	0	0	1	1
1	0	0	1	1	0	0	0	1	1
1	1	0	0	1	0	0	1	0	0

Basic Definitions

- **Duality Principle (DeMorgan's Theorem)**
- Verify DeMorgan's Theorem

$$x'y + xz'$$

$$= x'y + xz'$$

$$= ((x'y)' . (xz')')'$$

$$= ((x+y').(x'+z))'$$

Basic Definitions

- **Consensus Theorem**

$xy + x'z + yz = xy + x'z$

$(x+y)\cdot(x'+z)\cdot(y+z) = (x+y)\cdot(x'+z)$

Proof:

$xy + x'z + yz$

$= xy + x'z + 1.yz$

$= xy + x'z + (x+x')yz$

$= xy + x'z + xyz + x'yz$

$= (xy + xyz) + (x'z + x'zy)$

$= xy\,(1+z) + x'z\,(1+ y)$

$= xy + x'z$

Proof:

$(x+y)\bullet(x'+z)\bullet(y+z)$

$= (x+y)\bullet(x'+z)\bullet(0+y+z)$

$= (x+y)\bullet(x'+z)\bullet((xx')+y+z)$

$= (x+y)\bullet(x'+z)\bullet(x+y+z)\bullet(x'+y+z)$

$= ((x+y)+(0\bullet z))((x'+z)+(0\bullet y))$

$= (x+y)(x'+z)$

Operator Precedence

- The operator precedence for evaluating Boolean Expression is
  - Parentheses
  - NOT
  - AND
  - OR
- Examples
  - $x\ y' + z$
  - $(x\ y + z)'$

Boolean Functions

- Implementation with logic gates

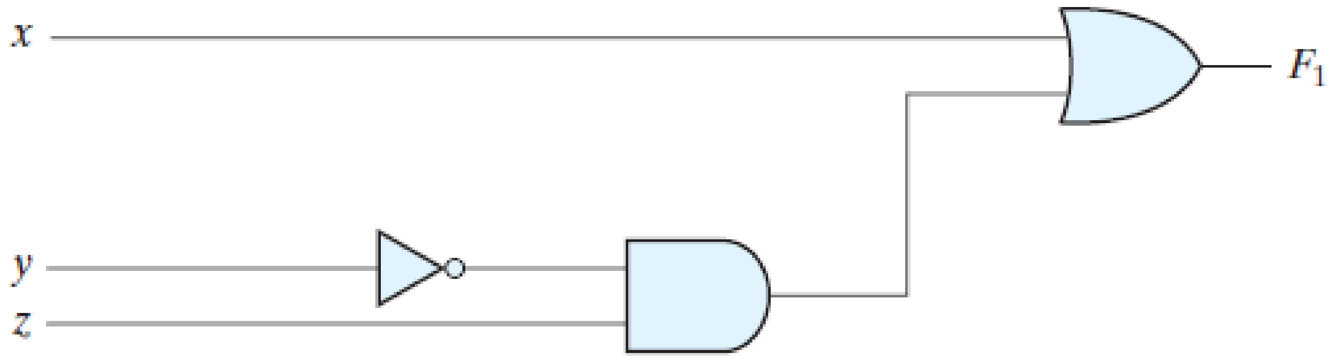


FIGURE 2.1
Gate implementation of $F_1 = x + y'z$

Boolean Functions

- Implementation with logic gates

Economical
Simpler ,
Less Cost
Fewer Gates
Fewer Inputs

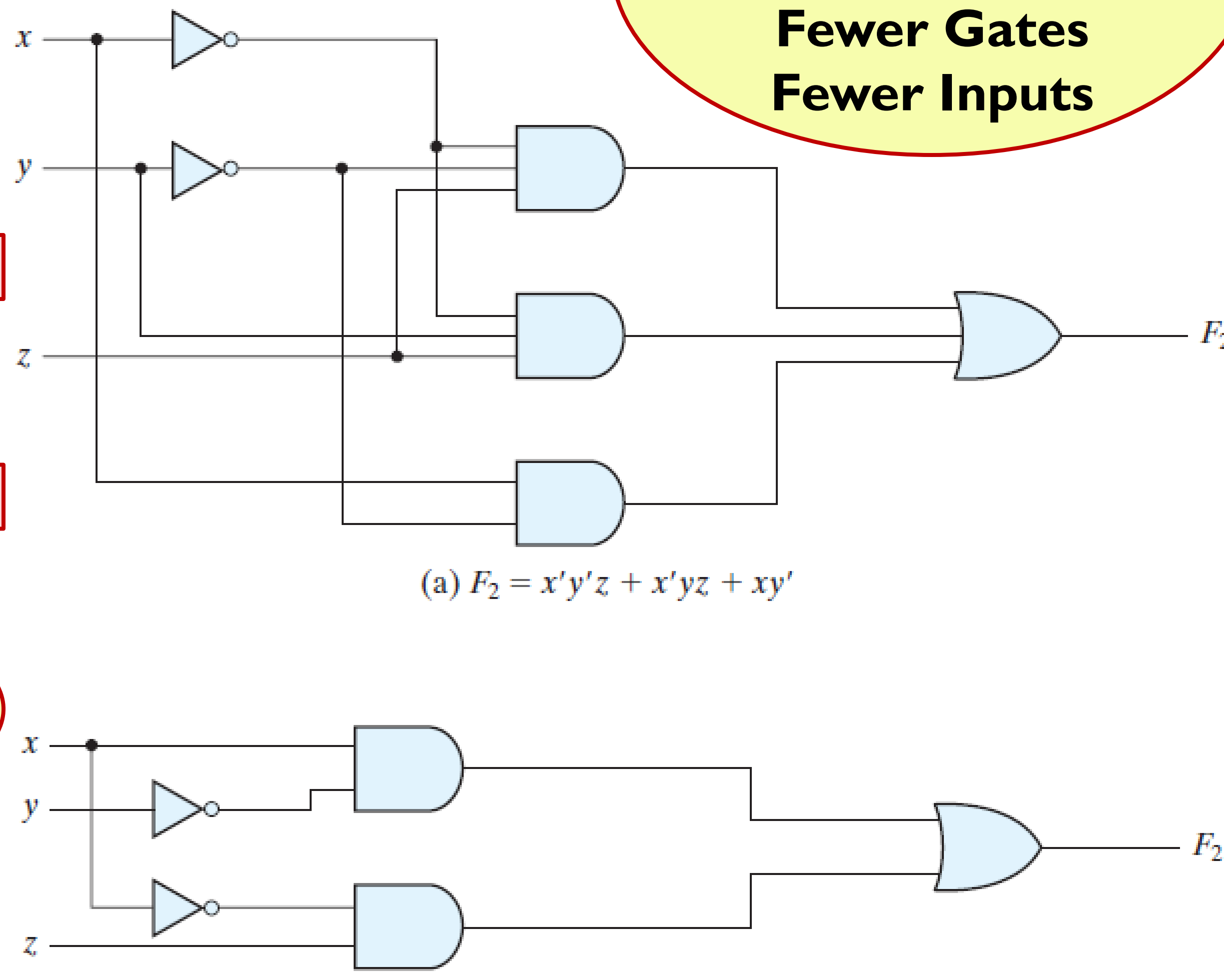
$F_3 = x'y'z + x'yz + xy'$

$= x'z(y'+y) + xy'$

$= x'z(1) + xy'$

$= x'z + xy'$

Simplification



(a) $F_2 = x'y'z + x'yz + xy'$

(b) $F_2 = xy' + x'z$

FIGURE 2.2
Implementation of Boolean function F_2 with gates

Complement of a Function

- The complement of a function F is F' and is obtained from an interchange of 0's for 1's and 1's for 0's in the value of F.

- The complement of a function may be derived algebraically through DeMorgan's theorems,
  - 3 variables DeMorgan's theorem
  - $(A+B+C)' = (A+X)'$ let $B+C = X$

$(A+B+C)' = (A+X)'$	let $B+C = X$
$= A'X'$	by theorem 5(a) (DeMorgan's)
$= A'(B+C)'$	substitute $B+C = X$
$= A'(B'C')$	by DeMorgan's theorem
$= A'B'C'$	by associative theorem

Complement of a Function

- The complement of a function *F* is *F'* and is obtained from an interchange of 0's for 1's and 1's for 0's in the value of *F*.

- Generalization: a function is obtained by interchanging AND and OR operators and complementing each literal.
  - F = $A+B+C+D+ \ldots$ *Then* $F' = (A+B+C+D+ \ldots)' = A'B'C'D' \ldots$
  - F = $ABCD \ldots$ *Then* $F' = (ABCD \ldots)' = A'+ B'+C'+D' \ldots$

Complement of a Function

- Find the Complement of the following functions
- $F_1 = x' y z' + x' y' z$
- $F_1' = (x'yz' + x'y'z)'$
  $= (x'yz')' (x'y'z)'$
  $= (x+y'+z) (x+y+z')$
- $F_2 = x(y' z' + y z)$
- $F_2' = [x(y'z'+yz)]'$
  $= x' + (y'z'+yz)'$
  $= x' + (y'z')' (yz)'$
  $= x' + (y+z) (y'+z')$
  $= x' + yz'+y'z$

Canonical and Standard Forms

Drive truth table of the following function

F = $x' y'$

x	y	x'	y'	F
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	0	0	0

Canonical and Standard Forms

Drive truth table of the following function

F = x' y' + x'y

x	y	x'	y'	$x'y'$	$x'y$	F
0	0	1	1	1	0	1
0	1	1	0	0	1	1
1	0	0	1	0	0	0
1	1	0	0	0	0	0

Canonical and Standard Forms

Drive truth table of the following function

F = x' y' + x'y + xy'

x	y	x'	y'	$x'y'$	$x'y$	xy'	F
0	0	1	1	1	0	0	1
0	1	1	0	0	1	0	1
1	0	0	1	0	0	1	1
1	1	0	0	0	0	0	0

Canonical and Standard Forms

Drive truth table of the following function

F = x'y' + x'y + xy' + xy

x	y	x'	y'	F
0	0	1	1	1
0	1	1	0	1
1	0	0	1	1
1	1	0	0	1

Canonical and Standard Forms

Drive truth table of the following function

F = x' y' z'

x	y	z	x'	y'	z'	F
0	0	0	1	1	1	1
0	0	1	1	1	0	0
0	1	0	1	0	1	0
0	1	1	1	0	0	0
1	0	0	0	1	1	0
1	0	1	0	1	0	0
1	1	0	0	0	1	0
1	1	1	0	0	0	0

Canonical and Standard Forms

Drive truth table of the following function

F = x' y' z' + x'y'z

x	y	z	x'	y'	z'	F
0	0	0	1	1	1	1
0	0	1	1	1	0	1
0	1	0	1	0	1	0
0	1	1	1	0	0	0
1	0	0	0	1	1	0
1	0	1	0	1	0	0
1	1	0	0	0	1	0
1	1	1	0	0	0	0

Canonical and Standard Forms

Minterms

- A minterm (standard product): an AND term consists of all literals in their normal form or in their complement form.

- For example,
- two binary variables *x* and *y*,
  - ***xy, xy', x'y, x'y'***
- It is also called a standard product.
- *n* variables can be combined to form 2^n minterms.

x	y	Term	Symbol
0	0	x'y'	m_0
0	1	x'y	m_1
1	0	xy'	m_2
1	1	xy	m_3

Minterms and Maxterms for Three Binary Variables

			Minterms	
x	***y***	***z***	**Term**	**Designation**
0	0	0	$x'y'z'$	m_0
0	0	1	$x'y'z$	m_1
0	1	0	$x'yz'$	m_2
0	1	1	$x'yz$	m_3
1	0	0	$xy'z'$	m_4
1	0	1	$xy'z$	m_5
1	1	0	xyz'	m_6
1	1	1	xyz	m_7

Canonical and Standard Forms

Minterms

- Sum of minterms for each combination of variables that produces a (1) in the function

$$\sum$$

Minterms and Maxterms for Three Binary Variables

			Minterms	
x	***y***	***z***	**Term**	**Designation**
0	0	0	$x'y'z'$	m_0
0	0	1	$x'y'z$	m_1
0	1	0	$x'yz'$	m_2
0	1	1	$x'yz$	m_3
1	0	0	$xy'z'$	m_4
1	0	1	$xy'z$	m_5
1	1	0	xyz'	m_6
1	1	1	xyz	m_7

Canonical and Standard Forms

Drive truth table of the following function

F = x' y'

x	y	x'	y'	F
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	0	0	0

F = $\sum m_0$

Canonical and Standard Forms

Drive truth table of the following function

F = x' y' + x'y

x	y	x'	y'	F
0	0	1	1	1
0	1	1	0	1
1	0	0	1	0
1	1	0	0	0

F = $\sum(m_0, m_1)$

F = x' y' + x'y
=(00,01)
=$\sum(m_0, m_1)$

Canonical and Standard Forms

Drive truth table of the following function

F = x'y' + x'y + xy' + xy

x	y	x'	y'	F
0	0	1	1	1
0	1	1	0	1
1	0	0	1	1
1	1	0	0	1

F = $\sum(m_0, m_1, m_2, m_3)$

Much More Compact Form
F = 1

Canonical and Standard Forms

Drive truth table of the following function

F = x' y' z' = $\sum(m_0)$

x	y	z	x'	y'	z'	F
0	0	0	1	1	1	1
0	0	1	1	1	0	0
0	1	0	1	0	1	0
0	1	1	1	0	0	0
1	0	0	0	1	1	0
1	0	1	0	1	0	0
1	1	0	0	0	1	0
1	1	1	0	0	0	0

Canonical and Standard Forms

Drive truth table of the following function

F = x' y' z' + x'y'z $= \sum(m_0, m_1)$

000, 001

x	y	z	x'	y'	z'	F
0	0	0	1	1	1	1
0	0	1	1	1	0	1
0	1	0	1	0	1	0
0	1	1	1	0	0	0
1	0	0	0	1	1	0
1	0	1	0	1	0	0
1	1	0	0	0	1	0
1	1	1	0	0	0	0

Canonical and Standard Forms

Write the following function in terms sum of its minterms

x	y	z	*F1*	*F2*	*F3*
0	0	0	1	0	1
0	0	1	1	1	1
0	1	0	0	0	0
0	1	1	0	0	0
1	0	0	0	1	0
1	0	1	0	0	1
1	1	0	0	0	1
1	1	1	1	0	1

F1 $= \sum(m_0, m_1, m_7)$

F1 = x'y'z' + x'y'z+ xyz

F2 $= \sum(m_1, m_4)$

F2 = x'y'z+ xy'z'

F3 $= \sum(m_{0,} m_1, m_5, m_6, m_7)$

Canonical and Standard Forms

Minterms and Maxterms

- A **minterm (standard product)**: an AND term consists of all literals in their normal form or in their complement form.

$$\sum$$

Sum means ORing

- A **maxterm (standard sums)**: an OR term
  - It is also called a standard sum.
  - 2^n maxterms.

$$\prod$$

Product means ANDing

Canonical and Standard Forms

Drive truth table of the following function

F = x + y

x	y	x'	y'	F
0	0	1	1	0
0	1	1	0	1
1	0	0	1	1
1	1	0	0	1

Canonical and Standard Forms

Drive truth table of the following function

F = (x + y) (x + y')

x	y	x'	y'	$x+y$	$x+y'$	F
0	0	1	1	0	1	0
0	1	1	0	1	0	0
1	0	0	1	1	1	1
1	1	0	0	1	1	1

Minterms and Maxterms

- Each maxterm is the complement of its corresponding minterm, and vice versa.

Table 2.3
Minterms and Maxterms for Three Binary Variables

			Minterms		Maxterms	
x	**y**	**z**	**Term**	**Designation**	**Term**	**Designation**
0	0	0	$x'y'z'$	m_0	$x + y + z$	M_0
0	0	1	$x'y'z$	m_1	$x + y + z'$	M_1
0	1	0	$x'yz'$	m_2	$x + y' + z$	M_2
0	1	1	$x'yz$	m_3	$x + y' + z'$	M_3
1	0	0	$xy'z'$	m_4	$x' + y + z$	M_4
1	0	1	$xy'z$	m_5	$x' + y + z'$	M_5
1	1	0	xyz'	m_6	$x' + y' + z$	M_6
1	1	1	xyz	m_7	$x' + y' + z'$	M_7

Minterms and Maxterms

Challenge

- Express the following functions in terms of
1. sum of standard product terms (minterms)
2. product of standard sum terms (Maxterms)

Table 2.4
Functions of Three Variables

x	*y*	*z*	Function f_1	Function f_2
0	0	0	0	0
0	0	1	1	0
0	1	0	0	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

F1= $\sum(m_1, m_4, m_7)$

= x'y'z + xy'z'+ xyz

= $\prod(M_0 M_2 M_3 M_5 M_6)$

=(x+y+z) (x+y'+z) (x+y'+z')

(x'+y+z') (x'+y'+z)

Minterms and Maxterms

Challenge

- Convert from any form to the other

Table 2.4
Functions of Three Variables

x	y	z	Function f_1	Function f_2
0	0	0	0	0
0	0	1	1	0
0	1	0	0	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

F1= $\sum(m_1, m_4, m_7)$

F1=x'y'z + xy'z'+ xyz

F1=(x+y+z) (x+y'+z) (x+y'+z')(x'+y+z') (x'+y'+z)

F1= $\prod(M_0 M_2 M_3 M_5 M_6)$

Minterms and Maxterms

- Express the following functions in terms of

1. sum of its minterms
2. product of its Maxterms

Table 2.4
Functions of Three Variables

x	y	z	Function f_1	Function f_2
0	0	0	0	0
0	0	1	1	0
0	1	0	0	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

F2= $\sum(m_3, m_5, m_6, m_7)$

= $\prod(M_0 M_1 M_2 M_4)$

Minterms and Maxterms

- Any Boolean function can be expressed as:
  - A sum of minterms expressions ("sum" meaning the ORing of terms)

$$F=XYZ+ XY'Z+ X'Y'Z'+ \ldots\ldots$$

  - A product of maxterms expressions ("product" meaning the ANDing of terms).

$$F=(X+Y+Z).(X+Y'+Z).(X'+Y'+Z') \ldots\ldots$$

  - Both Boolean functions are said to be in **Canonical** form.

Sum of Minterms

- Sum of minterms: there are 2^n minterms and 2^{2n} combinations of functions with *n* Boolean variables.
- Example : express $F = A+B'C$ as a sum of **minterms**.
  - $F = A+B'C$
    $= A(B+B') + B'C$
    $= AB + AB' + B'C$
    $= AB(C+C') + AB'(C+C') + (A+A')B'C$
    $= ABC+ABC'+AB'C+AB'C'+A'B'C$
    $= A'B'C + AB'C' + AB'C+ABC'+ ABC$
    $= m_1 + m_4 + m_5 + m_6 + m_7$
    $= \Sigma(1, 4, 5, 6, 7)$

1

2

or, built the truth table first

Table 2.5
Truth Table for $F = A + B'C$

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Sum of Minterms

- Sum of minterms: there are 2^n minterms and 2^{2n} combinations of functions with *n* Boolean variables.
- Example: express $F = A+B'C$ as a product of **maxterms**

2

  - $F = A+B'C$
    $= (A+B')(A+C)$
    $= (A+B'+CC')(A+C+BB')$
    $= (A+B'+C)(A+B'+C')(A+B+C)(A+B'+C)$
    $= \prod(M_0 M_2 M_3)$

1

or, built the truth table first

Table 2.5
Truth Table for $F = A + B'C$

A	*B*	*C*	*F*
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Product of Maxterms

- Product of maxterms: using distributive law to expand.
- Example : express $F = xy + x'z$ as a product of **maxterms**.
  - $F = xy + x'z$

$= (xy + x')(xy + z)$

$= (x+x')(y+x')(x+z)(y+z)$

$= (x'+y)(x+z)(y+z)$

1

$= (x'+y+zz')(x+z+yy')(y+z+xx')$

$=(x'+y+z)(x'+y+z')(x+z+y)\ (x+z+y')(y+z+x)$

$(y+z+x')$

$=(x+y+z)(x+y'+z)(x'+y+z)(x'+y+z')$

$= M_0M_2M_4M_5$

$= \Pi(0, 2, 4, 5)$

2

or, built the truth table first

Table 2.6
Truth Table for F = xy + x′z

x	*y*	*z*	*F*
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Complement of a Function Expressed in Canonical Forms

- The complement of a function expressed as the sum of minterms equals the sum of minterms missing from the original function.
  - $F(A, B, C) = \Sigma(1, 4, 5, 6, 7)$
  - $F(A, B, C) = = \Pi(0, 2, 3)$

Thus,

- $F'(A, B, C) = \Sigma(0, 2, 3)$
- $F'(A, B, C) = \Pi(1, 4, 5, 6, 7)$

- By DeMorgan's theorem $m_j' = M_j$

x	y	z	F1	F1'
0	0	0	0	1
0	0	1	1	0
0	1	0	0	1
0	1	1	0	1
1	0	0	1	0
1	0	1	1	0
1	1	0	1	0
1	1	1	1	0

Conversion between Canonical Forms

- To convert from one canonical form to another: **interchange** the symbols Σ and Π and list those numbers **missing** from the original form
  - Σ of 1's
  - Π of 0's

Conversion between Canonical Forms

- Example
  - $F = xy + x'z$
  - $F(x, y, z) = \Sigma(1, 3, 6, 7)$
  - $F(x, y, z) = \Pi(0, 2, 4, 6)$

Table 2.6
Truth Table for $F = xy + x'z$

x	*y*	*z*	*F*
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Canonical Forms vs. Standard Forms

Canonical Forms

- Each minterm or maxterm must contain **all the variables** either complemented or uncomplemented,
- Sum of minterms (Product terms)
- OR Product of Maxterms (sum terms)

Standard forms

- the terms that form the function may obtain **one, two, or any number** of literals, .
- There are two types of standard forms:
  - Sum of products:

    $F_1 = y' + xy + x'yz'$
  - Product of sums:

    $F_2 = x(y'+z)(x'+y+z')$

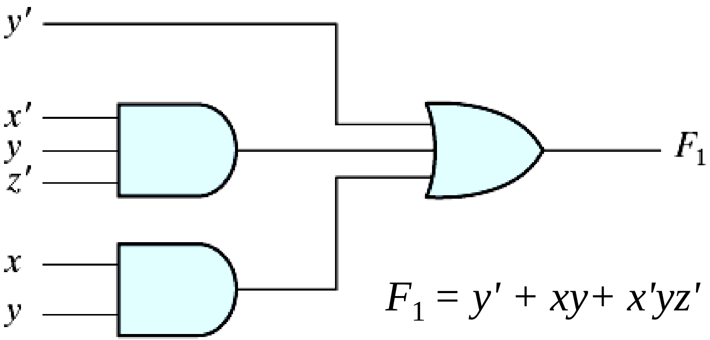
Standard Forms

- A Boolean function may be expressed in a **nonstandard** form
  - $F_3 = AB + C(B + A)$
- But it can be changed to a standard form by using The distributive law
  - $F_3 = AB + C(B + A) = AB + BC + AC$

- And it can be changed to a canonical form by using The distributive law after adding missing literal
  - $F_3 = AB + BC + AC = AB(C+C')+BC(A+A')+AC(B+B')$
  - $=ABC+ABC'+ABC+A'BC+ABC+AB'C$
  - $=ABC+ABC'+A'BC+AB'C$

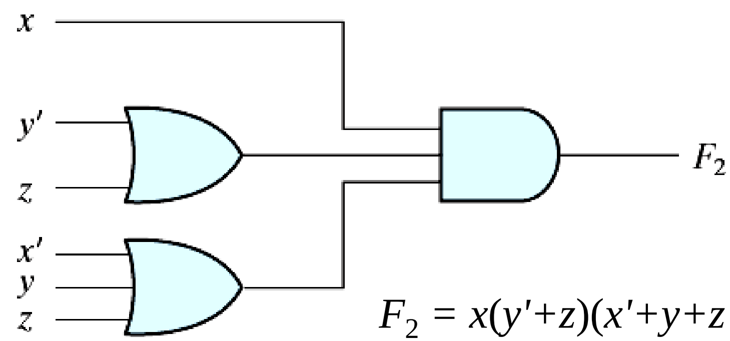
Implementation

- Two-level implementation



$F_1 = y' + xy + x'yz'$

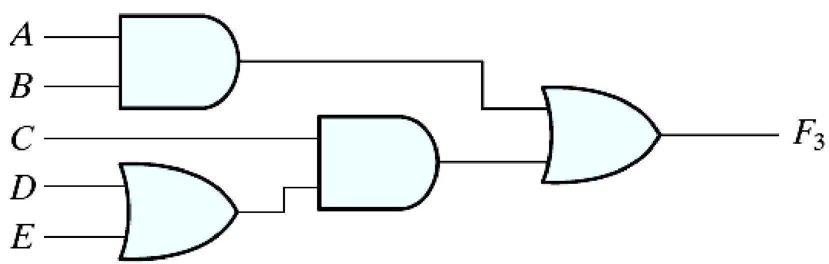
(a) Sum of Products



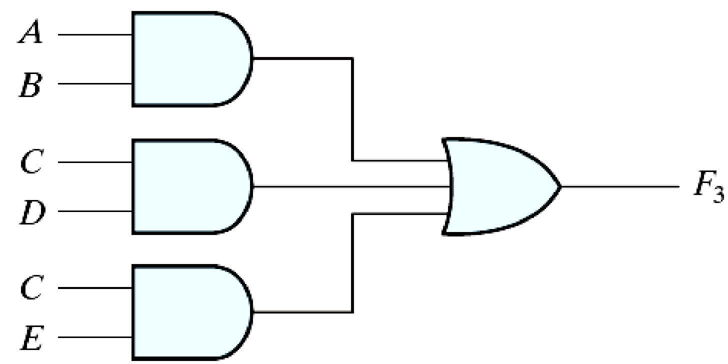
$F_2 = x(y'+z)(x'+y+z')$

(b) Product of Sums

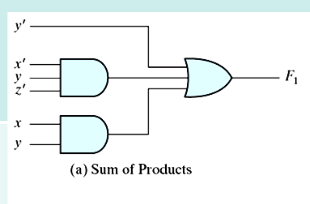
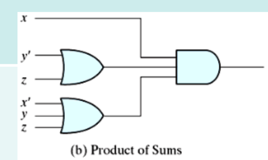
- Multi-level implementation



(a) $AB + C(D + E)$



(b) $AB + CD + CE$

SOP	POS
Sum of minterms $$F = \sum(m_0, m_2, \dots . m_i)$$ Sum of terms that function gives 1	Product of Maxterms $$F = \prod(M_0 M_1 \dots \dots . M_i)$$ Product of terms that function gives 0
Minterms (Locate 1's) m_0 = x‘y’z’ = 000 m_1 = x’y’z = 001 …. m_7 = xyz = 111	Maxterms (Locate 0's) M_0 = x+y+z = 000 M_1 = x+y+z’ = 001 …. M_7 =x‘+y+’z’ = 111
Convert Boolean function to SOP By multiplying each term by the missing variable Ored with its complement F = xy = xy(z+z’) = xyz +xyz’	Convert Boolean function to POS By expanding using distributive law and then for each term add the missing variable ANDed with its complement F= x+y = x+y+zz’ = (x+y+z)(x+y+z’)
 Logic Diagram: • 2 level implantation • Level of AND gates followed by one OR gate	 Logic Diagram: • 2 level implantation • Level of OR gates followed by one AND gate

Other Logic Operations

- 2^n rows in the truth table of n binary variables.
- 2^{2^n} functions for n binary variables.
- 16 functions of two binary variables.

Table 2.7
Truth Tables for the 16 Functions of Two Binary Variables

x	y	F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

- All the new symbols except for the exclusive-OR symbol are not in common use by digital designers.

Boolean Expressions

Table 2.8
Boolean Expressions for the 16 Functions of Two Variables

Boolean Functions	Operator Symbol	Name	Comments
$F_0 = 0$		Null	Binary constant 0
$F_1 = xy$	$x \cdot y$	AND	x and y
$F_2 = xy'$	x/y	Inhibition	x, but not y
$F_3 = x$		Transfer	x
$F_4 = x'y$	y/x	Inhibition	y, but not x
$F_5 = y$		Transfer	y
$F_6 = xy' + x'y$	$x \oplus y$	Exclusive-OR	x or y, but not both
$F_7 = x + y$	$x + y$	OR	x or y
$F_8 = (x + y)'$	$x \downarrow y$	NOR	Not-OR
$F_9 = xy + x'y'$	$(x \oplus y)'$	Equivalence	x equals y
$F_{10} = y'$	y'	Complement	Not y
$F_{11} = x + y'$	$x \subset y$	Implication	If y, then x
$F_{12} = x'$	x'	Complement	Not x
$F_{13} = x' + y$	$x \supset y$	Implication	If x, then y
$F_{14} = (xy)'$	$x \uparrow y$	NAND	Not-AND
$F_{15} = 1$		Identity	Binary constant 1


Thank You!