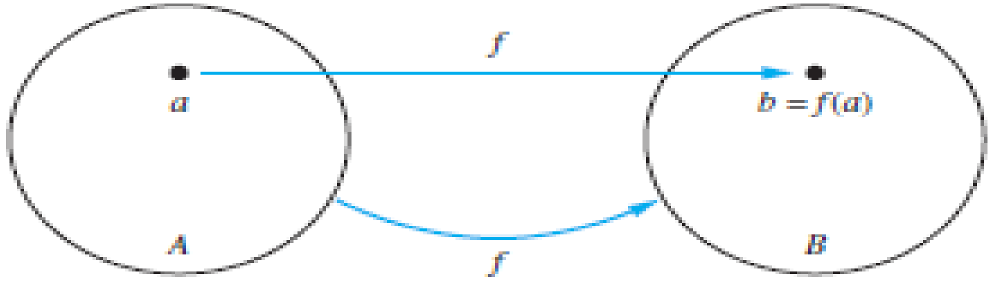
Functions

- Let *A* and *B* be nonempty sets. A *function f* from *A* to *B* ($f : A \to B$) is an assignment of exactly one element of *B* to each element of *A*.



We write $f(a) = b$ if *b* is the unique element of *B* assigned by the function *f* to the element *a* of *A*.

- If *f* is a function from *A* to *B*, we say that *A* is the *domain* of *f* and *B* is the *codomain* of *f.*
- If $f(a) = b$, we say that *b* is the *image* of *a* and *a* is a *preimage* of *b*.
- The *range*, or *image*, of *f* is the set of all images of elements of *A*. Also, if *f* is a function from *A* to *B*, we say that *f maps A* to *B*.

Examples

- Let $f : \boldsymbol{Z} \rightarrow \boldsymbol{Z}$ and $f(x) = x^2$

f (x) is function from Z to Z. because x^2 is defined for any integer number and well defined due to that the square of any integer number is unique.

- Let $f : \boldsymbol{R} \rightarrow \boldsymbol{R}$

a. $f(x) = 1/x$

f (x) is not function from R to R. because f(0) not defined.

b. $f(x) = \sqrt{x}$

f (x) is not function from R to R. because f(-1) not defined. In fact $\sqrt{x}$ *is not defined for any negative real number.*

c. $f(x) = \pm\sqrt{x^2 + 1}$

f (x) is not function from R to R. because it is not well defined. i.e some elements in the domain assigned to more than one element in the codomain.

d. $f(x) = x + 1$

f (x) is function from R to R.

Well defined function

- The function $f{:}A \to B$ is said to be well defined if and only if $\forall a\ \forall b\ (a = b \to f(a) = f(b))$, or equivalently (contrapositive) $\forall a\ \forall b\ (f(a) \neq f(b) \to a \neq b)$.

- The $f : \mathbf{Z} \to \mathbf{Z}$; $f(x) = x^2$ is well defined

Since if $x = y$ *then we can conclude that* $x^2 = y^2$. *Therefore* $f(x) = f(y)$.

- *The* $f : \mathbf{R} \to \mathbf{R}$; $f(x) = x + 1$ is well defined

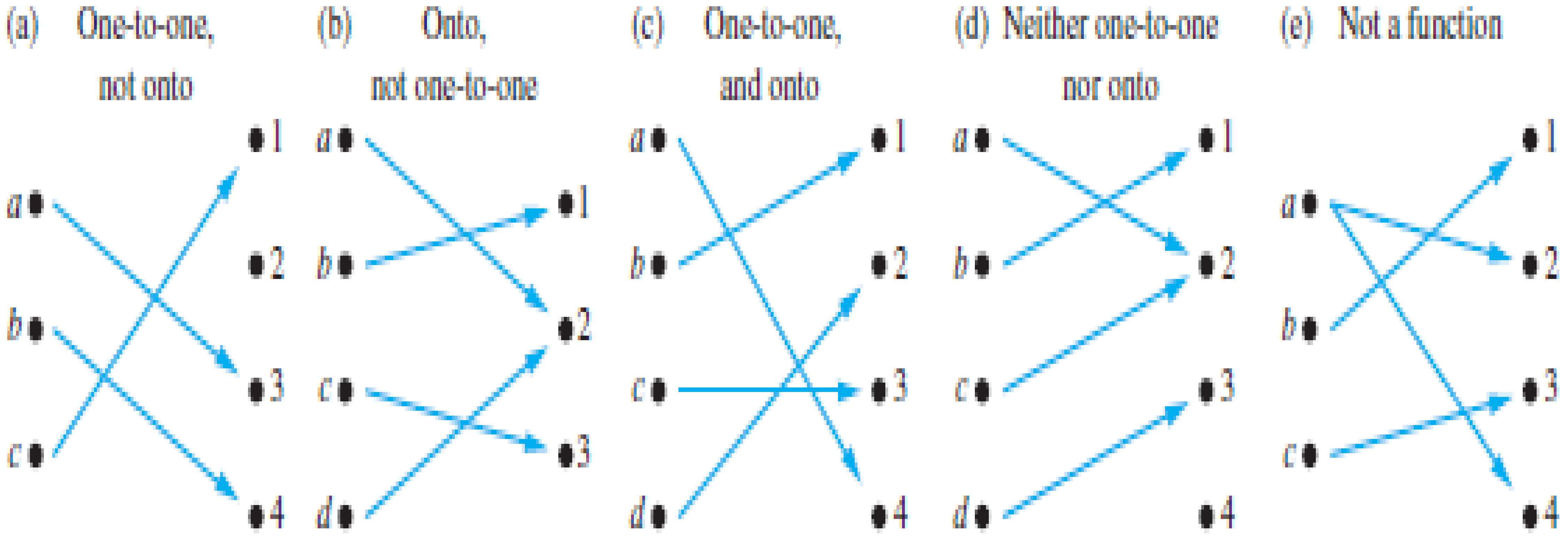
Since if $x = y$ *then* $x+1 = y+1$. *therefore* $f(x) = f(y)$.

one-to-one function (*injective*)

- The function $f{:}A \rightarrow B$ is said to be one-to-one (*injective*) function if and only if $\forall a\ \forall b\ (f(a) = f(b) \rightarrow a = b)$ or equivalently (contrapositive) $\forall a\ \forall b\ (a \neq b \rightarrow f(a) \neq f(b))$, where the universe of discourse is the domain of the function.
- Suppose that $f : A \rightarrow B$.
- *To show that f is injective* Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$, then $x = y$.
- *To show that f is not injective* Find particular elements $x, y \in A$ such that $x = y$ and $f(x) = f(y)$.

Onto function(*surjective*)

- A function f is onto (*surjective*) if $\forall y\, \exists x\, (f(x) = y)$, where the domain for x is the domain of the function and the domain for y is the codomain of the function.
- Suppose that $f : A \to B$.
- *To show that f is surjective* Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.
- *To show that f is not surjective* Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.


(a) One-to-one, not onto
(b) Onto, not one-to-one
(c) One-to-one, and onto
(d) Neither one-to-one nor onto
(e) Not a function
a
b
c
d
1
2
3
4

Determine whether each of these functions is an injection from ***R*** to ***R***.

a) $f(x) = 2x + 1$

Let $f(x) = f(y)$; $x, y \in \boldsymbol{R}$ $\therefore 2x + 1 = 2y + 1$

$\therefore 2x = 2y$ $\therefore x = y$

b) $f(x) = x^2 + 1$

Not 1-1 since f(-1) = f(1)=2 and -1 $\neq$ *1.*

c) $f(x) = (x^2 + 1)/(x^2 + 2)$

Not 1-1 since f(-1) = f(1)=2/3 and -1 $\neq$ *1.*

d) $f(x) = x^3$

Let $f(x) = f(y)$; $x, y \in \boldsymbol{R}$ $\therefore x^3 = y^3$ $\therefore x = y$

Determine whether each of these functions is a surjection from ***R*** to ***R***.

a) $f(x) = 2x + 1$

Let $y = f(x)$ $\therefore y = 2x + 1$ $\therefore 2x = y - 1$ $\therefore x = (y - 1)/2$

b) $f(x) = x^2 + 1$

Not onto since if *y = 0 then we can not find a preimage x (in the domain).*

c) $f(x) = (x^2 + 1)/(x^2 + 2)$

Not onto since if *y = -1 then we can not find a preimage x (in the domain).*

d) $f(x) = x^3$

Let $y = f(x)$ $\therefore y = x^3$ $\therefore x = \sqrt[3]{y}$

Since the cubic root is defined for all real number therefore we can get a preimage for all y (in the codomain).

- The function f is a *one-to-one correspondence,* if it is both one-to-one and onto. We also say that a function is *bijective*.

Determine whether each of these functions is a bijection from **R** to **R**.

a) $f(x) = 2x + 1$

The function is bijection.

b) $f(x) = x^2 + 1$

Not bijection due to its not 1-1.(or not onto)

c) $f(x) = (x^2 + 1)/(x^2 + 2)$

Not bijection due to its not onto.(or not 1-1)

d) $f(x) = x^3$

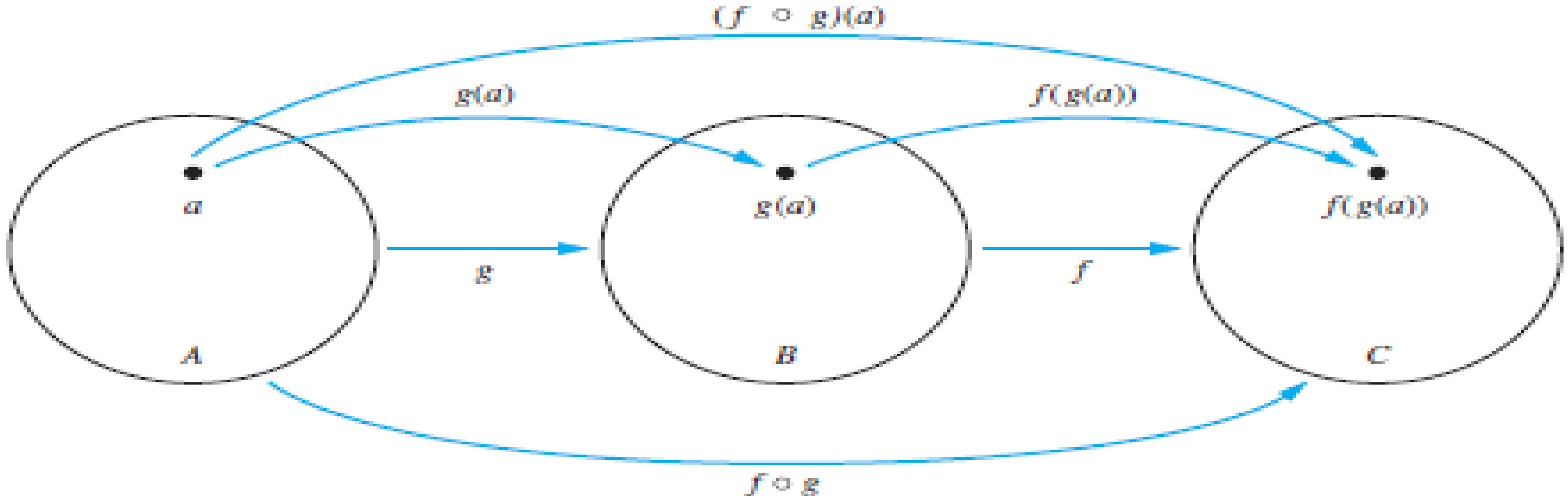
The function is bijection.

Increasing and decreasing functions

- A function f is increasing if $\forall x\ \forall y\ (x < y \rightarrow f(x) \leq f(y))$, strictly increasing if $\forall x\ \forall y\ (x < y \rightarrow f(x) < f(y))$,
- decreasing if $\forall x\ \forall y\ (x < y \rightarrow f(x) \geq f(y))$, and strictly decreasing if $\forall x\ \forall y\ (x < y \rightarrow f(x) > f(y))$, where the universe of discourse is the domain of f.

Composition of two functions

- Let g be a function from the set A to the set B and let f be a function from the set B to the set C.
- The *composition* of the functions f and g, denoted for all $a \in A$ by $f \circ g$, is defined by $(f \circ g)(a) = f(g(a))$.



Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g? What is the composition of g and f?

Solution:

Both the compositions $f \circ g$ and $g \circ f$ are defined. Moreover,

$(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$

and

$(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11.$

- The *identity function* is the function assigned the element to it self is donated by $I(x) = x$.
- The *inverse function* of $f: A \rightarrow B$ is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.
- The *composition* of the functions f and f^{-1} *donate the identity function, i.e* $f \circ f^{-1}(x) = f^{-1} \circ f(x) = I(x) = x$.
- *The function has inverse at whole codomain if and only if it is one-to-one and onto.(one-to-one correspondence / bijective)*

Find the inverse of the following function if possible ; f:$R \rightarrow R$

a) $f(x) = 2x + 1$

The function is bijection, therefore it has inverse.

Find inverse is same as finding preimage x (of domain) for the element y (of co-domain). Like check that the function is onto.

Let $y = f(x)$ $\therefore y = 2x + 1$ $\therefore 2x = y - 1$ $\therefore x = (y - 1)/2$

$\therefore f^{-1}(x) = (x - 1)/2$

b) $f(x) = x^2 + 1$

Not bijection , therefore has no inverse.

c) $f(x) = (x^2 + 1)/(x^2 + 2)$

Not bijection , therefore has no inverse.

d) $f(x) = x^3$

The function is bijection, therefore it has inverse.

Let $y = f(x)$ $\therefore y = x^3$ $\therefore x = \sqrt[3]{y}$ $\therefore f^{-1}(x) = \sqrt[3]{x}$