Sets

- A set is an unordered collection of objects, called elements or members of the set.
- We write $a \in A$ to denote that a is an element of the set A.
- The notation $a \in A$ denotes that a is not an element of the set A.
- Let A={-1,0,1,3}

Then $0 \in A$ but $2 \notin A$.

- Let B={1,{1},2}

Then $1 \in A$ and $\{1\} \in A$ but $\{2\} \notin A$ and $2 \in A$.

1. List the members of these sets.

a) $\{x \mid x$ is a real number such that $x^2 = 1\}$

b) $\{x \mid x$ is a positive integer less than 12$\}$

Solution: a) {-1 , 1}. b) {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}

2. Use set builder notation to give a description of each of these sets.

a) {0, 3, 6, 9, 12}

b) {−3, −2, −1, 0, 1, 2, 3}

c) {*a*, *e*, *i*, *o*, *u*}

Solution: a) $\{3x \mid x$ is a non negative integer less than 5$\}$

b) $\{x \mid x$ is an integer ; $-3 \leq x \leq 3\}$

c) $\{x \mid x$ is a vowel in the English alphabet$\}$

- We see that $A \subseteq B$ (A is a subset of B) if and only if $\forall x(x \in A \rightarrow x \in B)$. i.e every element of A is also an element of B.
- To show that $A \subseteq B$, show that if $x \in A$ then also $x \in B$.
- To show that $A \nsubseteq B$, find a single $x \in A$ such that $x \in B$.
- For every set S, (i) $\emptyset \subseteq S$

Because the empty set contains no elements, it follows that $x \in \emptyset$ is always false. It follows that the conditional statement $x \in \emptyset \rightarrow x \in S$ is always true, therefore $\forall x(x \in \emptyset \rightarrow x \in S)$ is true. #

Anther solution:

Let by contradiction $\emptyset \nsubseteq S$ i.e $\exists x(x \in \emptyset \land x \notin S)$ this is contradiction Because the empty set contains no elements. Therefore $\emptyset \subseteq S$.

- and (ii) $S \subseteq S$.

This is a trivial prove since when $x \in S$ then we can conclude that $x \in S$. Therefore $\forall x(x \in S \rightarrow x \in S)$. #

- Two sets A and B are equal ($A = B$) if and only if $\forall x(x \in A \leftrightarrow x \in B)$. i.e They have the same elements.
- To show that two sets A and B are equal, show that $A \subseteq B$ and $B \subseteq A$.
- Given a set S, the power set of S is the set of all subsets of the set S. The power set of S is denoted by P(S).
- What is the power set of the set {0, 1, 2}?

Solution: $P(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$.

Note that the empty set and the set itself are members of this set of subsets.

- What is the power set of the empty set? What is the power set of the set $\{\emptyset\}$?

Solution: $P(\emptyset) = \{\emptyset\}$.

$P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$.

- Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a finite set and that n is the cardinality of S. The cardinality of S is denoted by $|S|$.
- $|\{0, 1, 2\}|=3$, $|\emptyset|=0$, $|P(\{0, 1, 2\})|=8$
- If a set has n elements, then its power set has 2^n elements.

- Prove that P(A) ⊆ P(B) if and only if A ⊆ B.

Solution: 1^{st} : let P(A) ⊆ P(B) ∴ $\forall x(x \in P(A) \rightarrow x \in P(B))$

∵ A ∈ P(A) ∴ A ∈ P(B) ∵ P(B) is the set of all subsets of B

∴ A ⊆ B

2^{nd} : let A ⊆ B and Let S ∈ P(A)

∵ P(A) is the set of all subsets of A ∴ S ⊆ A ∴ S ⊆ B

∵ P(B) is the set of all subsets of B ∴ S ∈ P(B) ∴ P(A) ⊆ P(B) #

- The Cartesian product of two sets A and B is donated by

$A \times B = \{(a, b) \mid a \in A \land b \in B\}$; where (a, b) is ordered pair.

- What is the Cartesian product of A = {1, 2} and B = {a, b, c}?

Solution: $A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$.

Note that $A \times B$ not equal $B \times A$, unless $A = \emptyset$ or $B = \emptyset$ (so that $A \times B = \emptyset$) or $A = B$

- $A1 \times A2 \times \cdots \times An = \{(a1, a2,...,an) \mid ai \in Ai \text{ for } i = 1, 2,...,n\}$; where (a1, a2,...,an) is ordered n-tuples

Note that when A, B, and C are sets, $(A \times B) \times C$ is not the same as $A \times B \times C$

- The union of the sets A and is given by $A \cup B = \{x \mid x \in A \lor x \in B\}$.
- The intersection is given by $A \cap B = \{x \mid x \in A \land x \in B\}$.

Two sets are called disjoint if their intersection is the empty set.

- The difference of A and B is given by $A - B = \{x \mid x \in A \land x \notin B\}$.
- The complement of the set A is given by $\overline{A} = \{x \mid x \notin A\}$.

Note that $A - B = A \cap B$ and $\overline{A} = U - B$; U is the universal set.

Set Identities

Identity	Name
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Use set builder notation and logical equivalences to establish the first De Morgan law $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Solution:

$\overline{A \cap B} = \{x \mid x \notin A \cap B\}$	by definition of complement
$= \{x \mid \neg(x \in (A \cap B))\}$	by definition of does not belong symbol
$= \{x \mid \neg(x \in A \wedge x \in B)\}$	by definition of intersection
$= \{x \mid \neg(x \in A) \vee \neg(x \in B)\}$	by the first De Morgan law for logical equivalences
$= \{x \mid x \notin A \vee x \notin B\}$	by definition of does not belong symbol
$= \{x \mid x \in \overline{A} \vee x \in \overline{B}\}$	by definition of complement
$= \{x \mid x \in A \cup B\}$	by definition of union
$= \overline{A} \cup \overline{B}$	by meaning of set builder notation

Show that $\overline{A \cup (B \cap C)} = (\bar{C} \cup \bar{B}) \cap \bar{A}$, where A, B, and C are sets.

Solution:

$\overline{A \cup (B \cap C)} = \bar{A} \cap \overline{(B \cap C)}$ by the first De Morgan law

$= \bar{A} \cap (\bar{B} \cup \bar{C})$ by the second De Morgan law

$= (\bar{B} \cup \bar{C}) \cap \bar{A}$ by the commutative law for intersections

$= (\bar{C} \cup \bar{B}) \cap \bar{A}$ by the commutative law for unions.

Show that $A \cup (B - A) = A \cup B$ where A and B are sets.

- using set identities

$A \cup (B - A) = A \cup (B \cap \overline{A})$

$=(A \cup B) \cap (A \cup \overline{A})$

$=(A \cup B) \cap U = A \cup B$ #

- Using set builder notation

$A \cup (B - A) = \{x \mid x \in A \vee x \in B - A\}$

$= \{x \mid x \in A \vee (x \in B \wedge x \notin A)\}$

$= \{x \mid (x \in A \vee x \in B) \wedge (x \in A \vee x \notin A)\}$

$= \{x \mid (x \in A \vee x \in B) \wedge \mathrm{T}\}$

$= \{x \mid x \in A \vee x \in B\} = A \cup B$ #

- By proving $A \cup (B - A) \subseteq A \cup B$ *and* $A \cup B \subseteq A \cup (B - A)$.

1st $A \cup B \subseteq A \cup (B - A)$

Let $x \in A \cup B$ $\therefore x \in A \vee x \in B$

$\therefore (x \in A \vee x \in B) \wedge (x \in A \vee x \notin A)$

$\therefore x \in A \vee (x \in B \wedge x \notin A)$

$\therefore x \in A \vee x \in B - A$

$\therefore x \in A \cup (B - A)$ #

2nd $A \cup (B - A) \subseteq A \cup B$

Let $x \in A \cup (B - A)$

$\therefore x \in A \vee x \in B - A$

$\therefore x \in A \vee (x \in B \wedge x \notin A)$

$\therefore (x \in A \vee x \in B) \wedge (x \in A \vee x \notin A)$

$\therefore (x \in A \vee x \in B) \wedge \mathrm{T}$

$\therefore x \in A \vee x \in B$ $\therefore x \in A \cup B$ #

TABLE 2 A Membership Table for the Distributive Property.

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

What can you say about the sets A and B if we know that

- **a)** $A \cup B = A$?
- $B \subseteq A$
- **b)** $A \cap B = A$?
- $A \subseteq B$
- **c)** $A - B = A$?
- $A \cap B = \emptyset$
- **d)** $A \cap B = B \cap A$?
- *No thing*
- **e)** $A - B = B - A$?
- $B = A$