Mathematical induction

- Suppose that we have an infinite ladder, as shown in Figure 1, and we want to know whether we can reach every step on this ladder. We know two things:

1. We can reach the first rung of the ladder.

2. If we can reach a particular rung of the ladder, then we can reach the next rung.

- We conclude that we can reach every rung.

By (1), we know that we can reach the first rung of the ladder. Moreover, because we can reach the first rung, by (2), we can also reach the second rung; it is the next rung after the first rung. Applying (2) again, because we can reach the second rung, we can also reach the third rung. Continuing in this way, we can show that we can reach the fourth rung, the fifth rung, and so on……

- In the same manner to prove that $P(n)$ is true $\forall n \geq i$; *n is an* integer, where $P(n)$ is a propositional function, we complete two steps

1. Basis step

We verify that $P(i)$ is true.

2. Inductive step

We show that the conditional statement $P(k) \rightarrow P(k + 1)$ is true for all positive integers k.

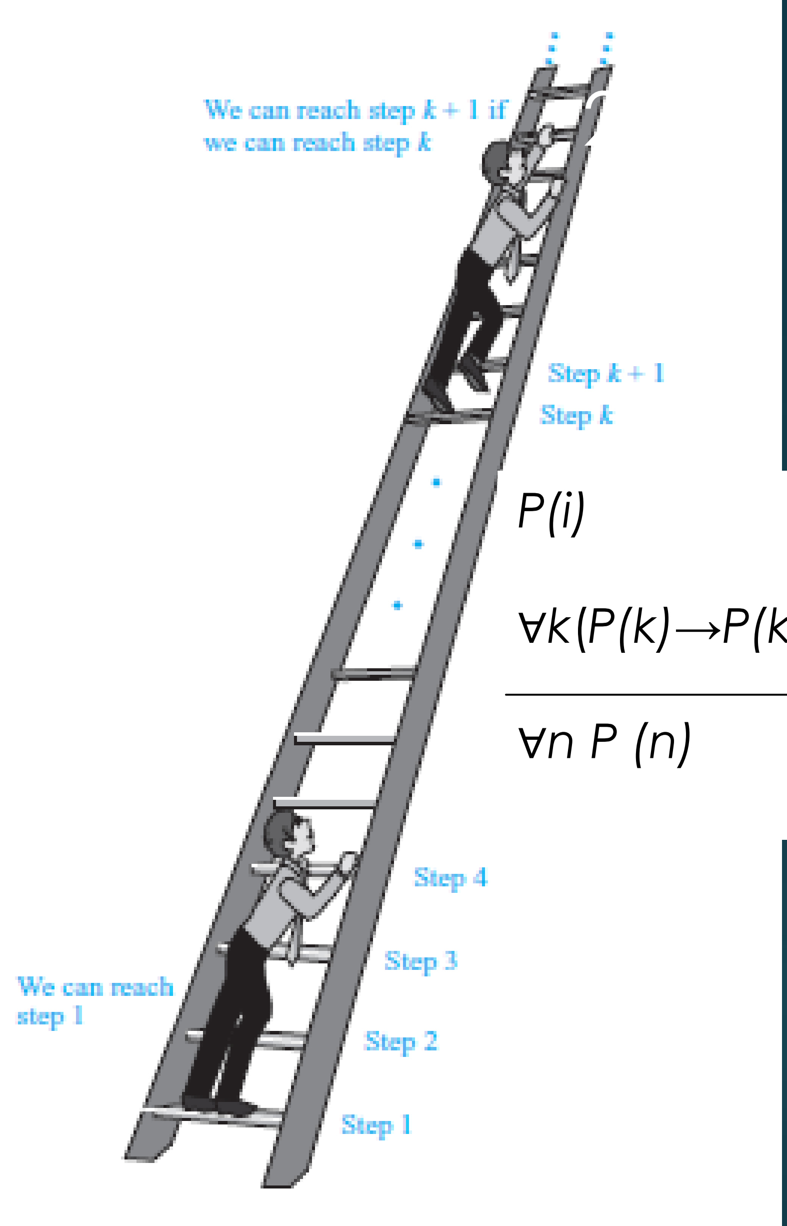

FIGURE 1 Climbing an Infinite Ladder.

$$\frac{P(i) \qquad \forall k(P(k) \rightarrow P(k+1))}{\forall n\, P(n)}$$

Template for Proofs by Mathematical Induction

- Express the statement that is to be proved in the form "for all $n \geq i$, $P(n)$" for a fixed integer i.
- Basis Step:

show that $P(i)$ is true, taking care that the correct value of (i) is used. This completes the first part of the proof.

- Inductive Step:

1.assume that $P(k)$ is true for an arbitrary fixed integer $k \geq i$. (inductive hypothesis)

2. State the statement $P(k + 1)$ which we need to be prove.

- Prove the statement $P(k + 1)$ making use the assumption $P(k)$.

5. Prove that

$$1^2+3^2+5^2+\cdots+(2n+1)^2=(n+1)(2n+1)(2n+3)/3$$

whenever n is a nonnegative integer.

- Let $P(n): 1^2+3^2+5^2+\cdots+(2n+1)^2=\frac{(n+1)(2n+1)(2n+3)}{3}$; $n \geq 0$
- Basis step: $P(0): 1^2=\frac{(0+1)(2\cdot 0+1)(2\cdot 0+3)}{3}$

The L.H.S=R.H.S=1 $\therefore P(0)$ is true.

- Inductive step:

1.assume that $P(k)$ Is true.

$$P(k): 1^2+3^2+5^2+\cdots+(2k+1)^2=\frac{(k+1)(2k+1)(2k+3)}{3} \quad (*)$$

2. we need to prove $P(k+1)$ is true?

$$P(k+1): 1^2+3^2+5^2+\cdots+(2(k+1)+1)^2=\frac{(k+2)(2k+3)(2k+5)}{3}$$

- From the L.H.S of $P(k+1)$:

$$1^2+3^2+5^2+\cdots+(2k+3)^2=\underline{1^2+3^2+5^2+\cdots+(2k+1)^2}+(2k+3)^2$$

Using $(*)$ we get

$$1^2+3^2+5^2+\cdots+(2k+3)^2=\frac{(k+1)(2k+1)(2k+3)}{3}+(2k+3)^2$$

$$=\frac{(2k+3)}{3}\big((k+1)(2k+1)+3\,(2k+3)\big)=\frac{(2k+3)}{3}(2k^2+3k+1+6k+9)$$

$$=\frac{(2k+3)}{3}(2k^2+9k+10)=\frac{(2k+3)}{3}(2k+5)(k+2)$$

Equal the R.H.S of $P(k+1)$.

Therefore $P(k+1)$ is true, by mathematical induction, $P(n)$ is true for all nonnegative integer n.

10. a) Find a formula for $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}$

b) For which nonnegative integers n the formula you conjectured in part (a) is valid? Prove your answer.

- a)

$$\frac{1}{1\cdot 2}=\frac{1}{2}$$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}=\frac{4}{6}=\frac{2}{3}$$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}=\frac{9}{12}=\frac{3}{4}$$

Therefore we can note that the formula is

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

- b) the formula valid for any positive integer.
- Let $P(n)$: $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$; $n\geq 1$
- Basis step: $P(1)$: $\frac{1}{1\cdot 2}=\frac{1}{1+1}$

The L.H.S=R.H.S=$\frac{1}{2}$ $\therefore P(1)$ is true.

- Inductive step:

1.assume that $P(k)$ Is true.

$$P(k): \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}=\frac{k}{k+1}\,(*)$$

2. we need to prove $P(k+1)$ is true?

$$P(k+1): \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}$$

- From the L.H.S of $P(k+1)$:

$$\underline{\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}}+\frac{1}{(k+1)(k+2)}$$

Using $(*)$ we get

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}=\frac{k(k+2)+1}{(k+1)(k+2)}=\frac{k^2+2k+1}{(k+1)(k+2)}=\frac{k+1}{k+2}$$

Equal the R.H.S of $P(k+1)$.

Therefore $P(k+1)$ is true, by mathematical induction, $P(n)$ is true for any positive integer n.

12. Prove that $\sum_{j=0}^{n}(-\frac{1}{2})^j = \frac{2^{n+1}+(-1)^n}{3\cdot 2^n}$ whenever *n* is a nonnegative integer.

- Let $P(n): \sum_{j=0}^{n}(-\frac{1}{2})^j = \frac{2^{n+1}+(-1)^n}{3\cdot 2^n}$; $n \geq 0$
- Basis step: $P(0): (-\frac{1}{2})^0 = \frac{2^{0+1}+(-1)^0}{3\cdot 2^0}$

The L.H.S=R.H.S=1 $\therefore P(0)$ is true.

- Inductive step:

1.assume that $P(k)$ Is true.

$$P(k): \sum_{j=0}^{k}(-\frac{1}{2})^j = \frac{2^{k+1}+(-1)^k}{3\cdot 2^k} \quad (*)$$

2. we need to prove $P(k+1)$ is true?

$$P(k+1): \sum_{j=0}^{k+1}(-\frac{1}{2})^j = \frac{2^{k+2}+(-1)^{k+1}}{3\cdot 2^{k+1}}$$

- From the L.H.S of $P(k+1)$:

$$\sum_{j=0}^{k+1}(-\frac{1}{2})^j = \sum_{j=0}^{k}(-\frac{1}{2})^j + (-\frac{1}{2})^{k+1}$$

Using $(*)$ we get

$$\sum_{j=0}^{k+1}(-\frac{1}{2})^j = \frac{2^{k+1} + (-1)^k}{3 \cdot 2^k} + (-\frac{1}{2})^{k+1} = \frac{2^{k+1} + (-1)^k}{3 \cdot 2^k} + \frac{(-1)^{k+1}}{2^{k+1}}$$

$$= \frac{2\left(2^{k+1} + (-1)^k\right) + 3(-1)^{k+1}}{3 \cdot 2^{k+1}}$$

$$= \frac{2^{k+2} + 2(-1)^k - 3(-1)^k}{3 \cdot 2^{k+1}}$$

$$= \frac{2^{k+2} - (-1)^k}{3 \cdot 2^{k+1}} = \frac{2^{k+2} + (-1)^{k+1}}{3 \cdot 2^{k+1}}$$

Equal the R.H.S of $P(k+1)$.

Therefore $P(k+1)$ is true, by mathematical induction, $P(n)$ is true for all nonnegative integer n.

34-Prove that 6 divides $n^3 - n$ whenever n is a nonnegative integer.

- Let $P(n)$: 6 divides $n^3 - n$; $n \geq 0$
- Basis step: $P(0)$: 6 divides $0^3 - 0$

$0^3 - 0 = 0$, $0/6 = 0$ $\therefore$ 6 divides $0^3 - 0$

$\therefore$ $P(0)$ is true.

- Inductive step:

1.assume that $P(k)$ Is true. $P(k)$: 6 divides $k^3 - k$ $(*)$

2. we need to prove $P(k+1)$ is true? $P(k+1)$: 6 divides $(k+1)^3 - (k+1)$

- $(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1 = k^3 - k + 3(k^2 + k)$

From $(*)$ we know that 6 divides $k^3 - k$. Also we note that $k^2 + k$ is an even integer, since $k^2 + k = k(k+1)$ is a product of Two consecutive integers which one of them must be even. So $k^2 + k = 2m$; m is an integer, thus 6 divides $3(k^2 + k)$. Therefore 6 divides $(k+1)^3 - (k+1)$.

Therefore $P(k+1)$ is true, by mathematical induction, $P(n)$ is true for all nonnegative integer n.

19. Let *P(n)* be the statement that $1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2}<2-\frac{1}{n}$,where *n* is an integer greater than 1.

- Let $P(n): 1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2}<2-\frac{1}{n}$; $n>1$
- Basis step: $P(2): 1+\frac{1}{4}<2-\frac{1}{2}$

The $L.H.S = 1\frac{1}{4} < R.H.S = 1\frac{1}{2}$ $\therefore P(2)$ is true.

- Inductive step:

1.assume that $P(k)$ Is true.

$$P(k): 1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{k^2}<2-\frac{1}{k} \quad (*)$$

2. we need to prove $P(k+1)$ is true?

$$P(k+1): 1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{(k+1)^2}<2-\frac{1}{k+1}$$

- From the L.H.S of $P(k+1)$:

$$1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{(k+1)^2}=\underline{1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{k^2}}+\frac{1}{(k+1)^2}$$

Using $(*)$ we get

$$1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{(k+1)^2}<2-\frac{1}{k}+\frac{1}{(k+1)^2}=2-(\frac{1}{k}-\frac{1}{(k+1)^2})$$

To prove $P(k+1)$ is true we will show that

$$2-\left(\frac{1}{k}-\frac{1}{(k+1)^2}\right)<2-\frac{1}{k+1}$$

To prove this inequality we just need to show that $\left(-\left(\frac{1}{k}-\frac{1}{(k+1)^2}\right)+\frac{1}{k+1}<0\right)$

$$-\left(\frac{1}{k}-\frac{1}{(k+1)^2}\right)+\frac{1}{k+1}=-\frac{1}{k}+\frac{1}{(k+1)^2}+\frac{1}{k+1}=\frac{-(k+1)^2+1+k(k+1)}{k(k+1)^2}$$

$$=\frac{-k^2-2k-1+1+k^2+k}{k(k+1)^2}=\frac{-k}{k(k+1)^2}=\frac{-1}{(k+1)^2}<0$$

Therefore $P(k+1)$ is true, by mathematical induction, $P(n)$ is true for all nonnegative integer n.

21. Prove that $2^n > n^2$ if n is an integer greater than 4.

- Let $P(n): 2^n > n^2\ ; n > 4$
- Basis step: $P(5): 2^5 > 5^2$

The $L.H.S = 2^5 = 32 > R.H.S = 5^2 = 25$ $\therefore P(5)$ is true.

- Inductive step:

1.assume that $P(k)$ Is true. $P(k): 2^k > k^2 \ (*)$

2. we need to prove $P(k+1)$ is true? $P(k+1): 2^{k+1} > (k+1)^2$

From $(*)$ we know that

$$2^k > k^2 \quad * 2$$

$$2 \cdot 2^k > 2 \cdot k^2$$

$$2^{k+1} > 2 \cdot k^2 = k^2 + k^2 = k^2 + \underbrace{(k + k + k + \cdots + k)}_{(k - times)} > k^2 + 2k + 1 > (k+1)^2$$

Therefore $P(k+1)$ is true, by mathematical induction, $P(n)$ is true for all integer n greater than 4.