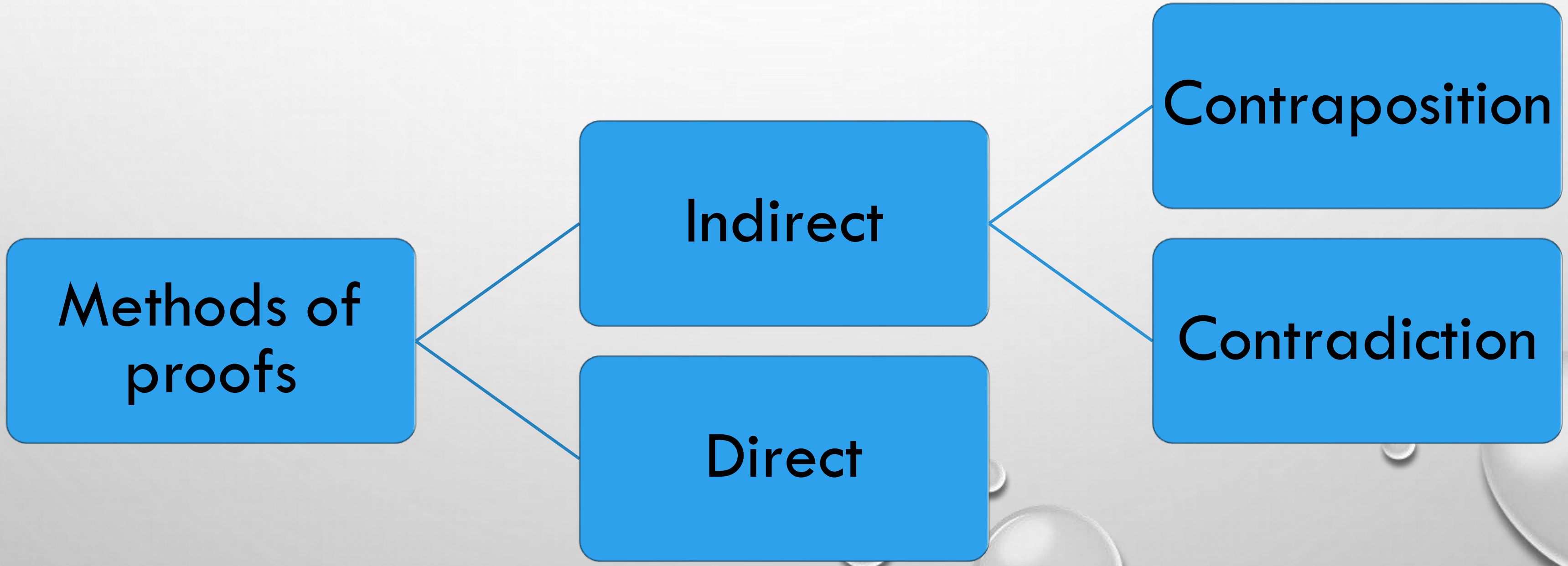
METHODS OF PROOFS

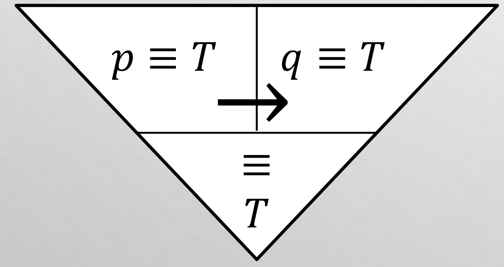


If we need to prove the statement $\forall x(P(x) \rightarrow Q(x))$ we can prove it using one of the following methods

- Direct proof
- $\forall x(P(x) \rightarrow Q(x))$
  $P(n)$ for any arbitrary n in the domain

$\therefore Q(n)$ for any arbitrary n in the domain

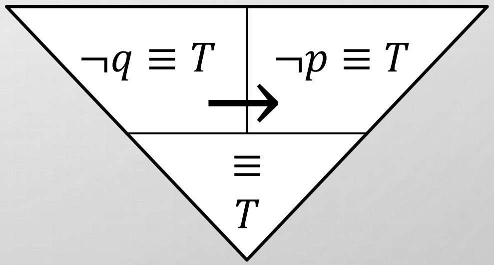
We start with the given conditions then we prove the results.



- Contraposition (indirDirect proof
- $\forall x(P(x) \rightarrow Q(x))$
  $\neg Q(n)$ for any arbitrary n in the domain

$\therefore \neg P(n)$ for any arbitrary n in the domain

We start with the negation of the results then we prove the given conditions is not satisfied.



CONTRADICTION

- To prove the statement p which we can not prove it with direct proof we will assume that the statement is false (i.e its negation is true)
- So we let $\neg p$ is true with assumption we expected that it will lead to a contradiction statement $(r \wedge \neg r)$ or contradiction with an axiom.

$\neg p \equiv T \to F \equiv F$ $\therefore \neg p \equiv F$ $\therefore p \equiv T$

that

7. Use a direct proof to show that every odd integer is the difference of two squares.

- Let $p(x)$: x is odd and $Q(x, y, z)$: x is the difference of squares of y and z. Where x, y, z are integers.

$$\forall x \left(p(x) \to \exists\, y\, \exists z\, Q(x, y, z)\right) \equiv \forall x\, \exists\, y\, \exists z \left(p(x) \to Q(x, y, z)\right); x, y, z \in Z$$

- Let $p(n)$ true for any integer n

$\therefore n = 2k + 1\,; k \in Z$ $\qquad \therefore n = 2k + 1 + (k)^2 - (k)^2$

$\because 2k + 1 + k^2 = (k + 1)^2$ $\qquad \therefore n = (k + 1)^2 - k^2$

$\therefore Q(n, k + 1, k)$ $\qquad for\ any\ n = 2k + 1\,; y = k + 1\,, z = k$

This is prove that every odd integer is the difference of two squares.

14. Prove that if x is rational and $x \neq 0$, then $\frac{1}{x}$ is rational.

- Let x is irrational and $x \neq 0$

$\therefore x = \frac{a}{b}$; $a, b \in Z$, $b \neq 0$

$\therefore \frac{1}{x} = \frac{b}{a}$; $b, a \in Z$; $a \neq 0$.

Since $x \neq 0$, therefore $a \neq 0$.

$\therefore \frac{1}{x}$ is rational number.

15. Use a proof by contraposition to show that if $x + y \geq 2$, where x and y are real numbers, then $x \geq 1$ or $y \geq 1$.

- Let $P(x, y): x + y \geq 2$ and $Q(x): x \geq 1$. where $x, y \in R$

$$\forall x \, \forall y; x, y \in R: \ (P(x, y) \rightarrow Q(x) \vee Q(y))$$

- Let by contraposition $x < 1$ and $y < 1$

$\therefore x + y < 1 + 1$

$\therefore x + y < 2$

Therefore by contraposition if $x + y \geq 2$, then $x \geq 1$ or $y \geq 1$.

13. Prove that if x is irrational, then $\frac{1}{x}$ is irrational.

- Let by contraposition if $\frac{1}{x}$ is rational, then x is rational

$\because \frac{1}{x}$ is rational $\quad\quad \therefore \frac{1}{x} = \frac{a}{b}$; $a, b \in Z$, $b \neq 0$

$\therefore x = \frac{b}{a}$; $b, a \in Z$; $a \neq 0$.

Since if $a = 0$ we note that $\frac{1}{x} = \frac{0}{b}$, therefore $1 = 0$ which is false.

$\therefore x$ is rational number.

$\therefore$ by contraposition if x is irrational, then $\frac{1}{x}$ is irrational.

8. Prove that if n is a perfect square, then $n + 2$ is not a perfect square.

- Let n is a perfect square i.e n is a square of an integer k.

$\therefore n = k^2$ so the next perfect square number is $(k+1)^2$

$\because (k+1)^2 - k^2 = 2k + 1$ $\because$ 1 is the minimum perfect square

$\therefore k \geq 1$ $\therefore (k+1)^2 - k^2 \geq 3$

$\therefore$ The difference between two perfect squares at least 3,

$\therefore n + 2$ is not a perfect square.

If $a, b \in Z$, then $a^2 - 4b \neq 2$.

- Let by contradiction $a^2 - 4b = 2$

$\therefore a^2 = 4b + 2 = 2(2b + 1)$

$\therefore a^2$ is even so we can write $a^2 = 2k$; $k \in Z$

$\therefore (2k)^2 = 2(2b + 1)$ $\therefore 4k^2 = 2(2b + 1)$

$\therefore 2k^2 = 2b + 1$

Since the L.H.S is an even integer and R.H.S is an odd integer, we get a contradiction with our assumption, Therefore $a^2 - 4b \neq 2$.

31. Show that these statements about the integer x are equivalent: (*i*) $3x + 2$ is even, (*ii*) $x + 5$ is odd, (*iii*) x^2 is even.

- We need to prove that *i* ≡ *ii* ≡ *iii* (the ≡ means ↔), so we need to prove *i* ↔ *ii* ≡ (*i* → *ii*) ∧ (*ii* → *i*) and also for (*ii* , *iii*) and (*iii* , *i*) Instead of this we prove that (*i* → *ii*) ∧ (*ii* → *iii*) ∧ (*iii* → *i*)

- So we 1st will prove *i* → *ii* (if $3x + 2$ is even, then $x + 5$ is odd)

Let by contraposition $x + 5$ is even $\therefore x + 5 = 2k\,; k \in Z \quad \therefore x = 2k - 5$

$\therefore 3x + 2 = 3(2k - 5) + 2 = 6k - 15 + 1 + 1 = 6k - 14 + 1 = 2(3k - 7) + 1$

$\therefore 3x + 2 = 2\,r\,; r = 3k - 7 \in Z \quad \therefore 3x + 2$ is odd #

- 2nd we will prove *ii* → *i* ($x + 5$ is odd , then x^2 is even)

Let $x + 5$ is odd $\therefore x + 5 = 2k + 1\,; k \in Z \;\therefore x = 2k - 4 = 2(k - 2) = 2\,l\,; l \in Z$

$\therefore x^2 = 4\,l^2 = 2\,(2l^2)$ which is even #

- Finally we will prove *iii* → *i* (x^2 is even , then $3x + 2$ is even)
- We can not prove this statement using direct prove or contraposition so we will true the contradiction but we have a conditional statement

$$\neg\left(\forall x\left(p(x) \rightarrow Q(x)\right)\right) \equiv \exists x\left(p(x) \wedge \neg Q(x)\right)$$

So we let x^2 is even and $3x + 2$ is odd

$\because 3x + 2 = 2k + 1 \,; k \in Z$ $\qquad\qquad \therefore 3x = 2k - 1$

$\because x^2 = 2\,l \,; l \in Z$ $\qquad\qquad \therefore 9x^2 = 4\,k^2 - 4k + 1$

$\therefore 2(9\,l) = 2(2\,k^2 - 2\,k) + 1$

$\therefore 2\,l_1 = 2\,k_1 + 1 \quad ; 9\,l = l_1 , 2\,k^2 - 2k = k_1 \ \in Z$

This is contradiction, therefore $3x + 2$ must be even.

11. Prove or disprove that the product of two irrational numbers is irrational.

- We will disprove this statement by counter example

Let $x = \sqrt{2}$ and $\mathrm{y} = \sqrt{8}$

$\therefore x \cdot y = \sqrt{2} \cdot \sqrt{8} = \sqrt{16} = 4$ which is rational number.

12. Prove or disprove that the product of a nonzero rational number and an irrational number is irrational.

- Let by contradiction r is a non zero rational number and x is irrational number and its product is rational

$\therefore r = \frac{a}{b}\ ; a, b\ \in Z\ , b \neq 0$

$\therefore r \cdot x = \frac{a}{b} \cdot x = \frac{c}{d}$

$\therefore x = \frac{b \cdot c}{a \cdot d} = \frac{m}{n}\quad ; m = b \cdot c \neq 0\ , n = a \cdot d \neq 0\ \ ; m, n \in Z$

$\therefore$ x is rational number this contradiction with the assumption x is rational.

$\therefore$ the product of a nonzero rational number and an irrational number is irrational.