Proving Theorems

Proof

- Direct proof: $p \to q$, Use True cases
- indirect proof
  - Proof by Contrapositive: $p \to q \equiv \neg q \to \neg p$
  - Proof by contradiction: "$p \to q$" $p \wedge \neg q \to F$

1. direct

Ex. prove That if $m+n$ & $n+p$ are even integers where m, n, p are integer Then $m+p$ is even. Proof by direct?

Sol.

$m+n = 2k \quad k \in \mathbb{Z} \quad \& \; m, n, p \in \mathbb{Z}$

$n+p = 2R \quad R \in \mathbb{Z}$

$m+n+n+p = 2k+2R = 2(k+R)$

$\Rightarrow m+2n+p = 2(k+R) \Rightarrow m+p = 2(k+R)-2n$

$= 2(k+R-n)$, $k, R, n \in \mathbb{Z}$

$\Rightarrow \therefore m+p$ is even.

Ex use direct Proof to show that. every odd integer is The difference of two squares. where $\Rightarrow$ $R^2 - S^2 = 2k+1$?? لازم اثبت ده

odd integer is $2k+1 = 2k+1+k^2-k^2$

$= k^2+2k+1-k^2 = (k+1)^2 - k^2 = R^2 - S^2$

Ex Prove if n is positive integer Then n is even iff $7n+4$ is even

$n \in \mathbb{Z}^+$, n even $\Longleftrightarrow$ $7n+4$ even.

"$\Rightarrow$" "$n = 2k$" is even $\Rightarrow 7n+4 = 7(2k)+4 = 2(7k)+4$

$= 2(7k+2) = 2 \times \square$

$\Rightarrow \therefore 7n+4$ is even.

"$\Leftarrow$" $7n+4 = 2k \Rightarrow 7n = 2k-4$

$\Rightarrow n+6n = 2k-4$

$\Rightarrow n = 2k-6n-4$

$= 2(k-3n-2) \Rightarrow \therefore n$ is even.

...w That. These statement about The integer x are equivalent

i) $3x+2$ is even. ii) $x+5$ is odd. iii) x^2 is even.

Proof by Contrapositive. $\neg q \to \neg p$ "iii → i"

" x^2 is even $\Rightarrow 3x+2$ is even "

by Contrapositive let $3x+2$ is odd $\Rightarrow 3x+2 = 2t+1$

$\Rightarrow x = 2t-1-2x = 2(t-x-1)+1 = 2s+1$ ∴ x is odd

$\Rightarrow x^2$ is odd no " where Product of two odd no is odd"

∴ x^2 is even $\Rightarrow 3x+2$ is even. "i → ii"

" $\overset{p}{3x+2}$ is even $\Rightarrow \overset{q}{x+5}$ is odd " by contrapositive

let $x+5$ is even $\Rightarrow x+5 = 2k$

$\Rightarrow x = 2k-5 = 2(k-3)+1 \Rightarrow$ ∴ x is odd

$\Rightarrow 3x+2 = 3(2(k-3)+1)+2 = 6k-18+3+2 = 6k-13 = 6k-14+1$

$= 2(3k-7)+1$

$\Rightarrow$ ∴ $3x+2$ is even $\Rightarrow x+5$ is odd.

Prove That. if x is irrational, Then $1/x$ is irrational? "$p \to q$"

let $1/x$ is rational $\Rightarrow 1/x = \frac{r}{q}$; $r, q \in \mathbb{Z}$, $q \neq 0$

$x = \frac{1}{1/x} = q/r \Rightarrow x$ is rational $\Rightarrow$ ∴ x irrational $\Rightarrow 1/x$ is irrational

Show that, if $n \in \mathbb{Z}$ & n^3+5 is odd Then n is even.

Proof by a) Contrapositive b) Contradiction.

a) $n \in \mathbb{Z}$, $\underset{p}{n^3+5}$ is odd $\Rightarrow \underset{q}{n \text{ is even}}$

assume n is odd

$\Rightarrow n = 2k+1$

$n^3+5 = (4k^2+4k+1)(2k+1)+5$

$= 4k^2+4k+1+8k^3+8k^2+2k+5$

$= 2(2k^2+2k+4k^3+4k^2+k+3)$

$= 2 \cdot S \Rightarrow$ ∴ n^3+5 is even.

∴ n^3+5 is odd $\Rightarrow n$ is even.

b) n^3+5 is odd "p" $\Rightarrow n$ is even q"

$\Rightarrow p \wedge \neg q \to F$.

$\Rightarrow n = 2k+1$ odd $\wedge$ n^3+5 is odd

n^2 is odd no

$\Rightarrow n^3$ is odd no

$\Rightarrow n^3+5$ is odd ?? False

" where n^3+5 is even if n is odd"

∴ n^3+5 is odd $\Rightarrow n$ is even.

• Find The Counter example. to statement That every $\mathbb{Z}^+$ Can be written as The sum of The squares of Three integers.

Soln $n \in \mathbb{Z}^+$ $n = k^2+r^2+t^2$; $k, r, t \in \mathbb{Z}$

Counter example $n = 15$. ∴ This statement is false.

⇒ Mathematical induction:-

- The statement $P(n)$ for all $n \geq i$
- **Basis step:** show that $P(i)$ is True.
- **Inductive step:** 1. assume that $P(k)$ is True.
  2. state the statement $P(k+1)$ which we need to be prove.

Prove That. $1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3$, for all $n \geq 0$.

Proof. let $P(n): 1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$; $n \geq 0$

. **Basis step:** $P(0) = 1^2 = \frac{(0+1)(2.0+1)(2.0+3)}{3}$

$\Rightarrow L.H.S = R.H.S = 1 \Rightarrow \therefore P(0)$ is True.

. **inductive step:** assume $P(k)$ is True.

$P(k) = 1^2 + 3^2 + 5^2 + \cdots + (2k+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}$ $\underline{(*)}$

we need prove $P(k+1)$ is True

$P(k+1): 1^2 + 3^2 + \cdots + (2(k+1)+1)^2 = \frac{(k+2)(2k+3)(2k+5)}{3}$

from L.H.S of $P(k+1)$

$1^2 + 3^2 + 5^2 + \cdots + (2k+3)^2 = 1^2 + 3^2 + 5^2 + \cdots + (2k+1)^2 + (2k+3)^2$

by use $(*)$.

$1^2 + 3^2 + 5^2 + \cdots + (2k+3)^2 = \frac{(k+1)(2k+1)(2k+3)}{3} + (2k+3)^2$

$= \frac{(2k+3)}{3}((k+1)(2k+1) + 3(2k+3))$

$= \frac{(2k+3)}{3}[2k^2 + 3k + 1 + 6k + 9] = \frac{(2k+3)}{3}(2k^2 + 9k + 10)$

$= \frac{(2k+3)}{3}(2k+5)(k+2) = \frac{(k+2)(2k+3)(2k+5)}{3} =$ R.H.S of $P(n)$

$\therefore P(k+1)$ is True $\Rightarrow P(n)$ is True for all $n \geq 0$.

a) find formula for $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\dots+\frac{1}{n(n+1)}$.

b) for nonnegative integers "n" The formula is Valid?

Proof $\frac{1}{1.2}=\frac{1}{2}$

$\frac{1}{1.2}+\frac{1}{2.3} = 4/6 = 2/3$ $\Rightarrow \therefore$ formula is

$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4} = 9/12 = 3/4 \Rightarrow \frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\dots+\frac{1}{n(n+1)} = \frac{n}{n+1}$; $n \geq 1$.

(b) Let $P(n): \frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\dots+\frac{1}{n(n+1)} = \frac{n}{n+1}$; $n \geq 1$

Basis: $P(1) = \frac{1}{1.2} = \frac{1}{1+1} = \frac{1}{2}$.

$\Rightarrow \therefore P(1)$ is True.

inductive: $P(k) = \frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\dots+\frac{1}{k(k+1)} = \frac{k}{k+1}$ "$\star$"

we need prove that. $P(k+1)$ is True.

$P(k+1) = \frac{1}{1.2}+\frac{1}{2.3}+\dots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$.

from $\star$

$= \frac{k}{k+1}+\frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)}$

$= \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{k+1}{k+2}$. $\Rightarrow \therefore$ L.H.S = R.H.S.

$\therefore P(k+1)$ is True.

$\Rightarrow \therefore P(n)$ is True for any +ve integer n.

Prove That. $2^n > n^2$, $n > 4$

let $P(n): 2^n > n^2$; $n > 4$

$\Rightarrow P(5) = 2^5 > 5^2$

assume ~~that~~ $P(k)$ is True

$2^k > k^2$ $\star$

we need proof $P(n)$ is true

must proof $P(k+1)$

$2^{k+1} > (k+1)^2$

$\Rightarrow 2^{k+1} > 2.k^2 = k^2+k^2$

$= k^2 + (k+k+k+\dots+k) > k^2+2k+1$

$> (k+1)^2$

$\therefore P(k+1)$ True

$\Rightarrow P(n)$ is True.

Prove That. $\sum_{j=0}^{n} (-\frac{1}{2})^j = \frac{2^{n+1} + (-1)^n}{3 \cdot 2^n}$ whenever n is a +ve integer.

Proof let $P(n) = \sum_{j=0}^{n} (-1/2)^j = \frac{2^{n+1} + (-1)^n}{3 \cdot 2^n}$; $n \geq 0$

① $P(0) = (-1/2)^0 = \frac{2^{0+1} + (-1)^0}{3 \cdot 2^0} = 3/3 = 1$.

$\therefore P(0)$ is true.

② assume That $P(k)$ is True

$$P(k): \sum_{j=0}^{k} (-1/2)^j = \frac{2^{k+1} + (-1)^k}{3 \cdot 2^k}$$

③ $P(k+1)$: $\sum_{j=0}^{k+1} (-1/2)^j = \sum_{j=0}^{k} (-1/2)^j + (-1/2)^{k+1} = \frac{2^{k+2} + (-1)^{k+1}}{3 \cdot 2^{k+1}}$

$$\Rightarrow \sum_{j=0}^{k+1} (-1/2)^j = \frac{2^{k+1} + (-1)^k}{3 \cdot 2^k} + (-1/2)^{k+1}$$

$$= \frac{2^{k+1} + (-1)^k}{3 \cdot 2^k} + \frac{(-1)^{k+1}}{2^{k+1}}$$

$$= \frac{\cancel{2^{k+1}}(2^{k+1} + (-1)^k) + 3 \cdot \cancel{2^k}(-1)^{k+1}}{3 \cdot \cancel{2^k} \cdot 2^{k+1}} = \frac{2(2^{k+1} + (-1)^k) + 3(-1)^{k+1}}{3 \cdot 2^{k+1}}$$

$$= \frac{2^{k+2} + 2(-1)^k - 3(-1)^k}{3 \cdot 2^{k+1}} = \frac{2^{k+2} - (-1)^k}{3 \cdot 2^{k+1}}$$

$$= \frac{2^{k+2} + (-1)^{k+1}}{3 \cdot 2^{k+1}}$$

$\therefore P(k+1)$ is True.

Scanned with CamScanner

Prove That. 6 divides $n^3 - n$ whenever n is a non negative integer.

<u>Proof</u> Let $P(n)$: 6 divides $n^3 - n$; $n \geq 0$

$P(0)$: 6 divides $0^3 - 0 \Rightarrow 0/6 = \underline{0}$.

$\Rightarrow P(0)$ True

assume $P(k)$ is True

$\Rightarrow P(k)$: 6 divides $k^3 - k$.

$\Rightarrow P(k+1)$: 6 divides $(k+1)^3 - (k+1)$

$\Rightarrow (k+1)^3 - (k+1) = k^3 + 3k^2 + 3k \cancel{+1} - k \cancel{-1}$

$= k^3 - k + 3(k^2 + k)$

— we know That 6 divides $k^3 - k$.

& also $k^2 + k = k(k+1)$ is even

$\Rightarrow 3(k^2 + k)$ is even

$\Rightarrow$ 6 divided $3(k^2 + k)$.

$\therefore$ 6 divided $(k+1)^3 - (k+1)$.