CH.1 The Foundations: Logic and Proofs

II. predicate logic

- predicate logic
- The Language of Quantifiers
    - Logical Equivalences Involving Quantifiers
    - Nested Quantifiers
    - Rules of Inference

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• predicate logic

In this section we will introduce a more powerful type of logic called **predicate logic**

Propositional logic cannot adequately express the meaning of all statements in mathematics and in natural language. For example,

- "Every computer connected to the university network is functioning properly."

Statements involving variables, such as

- "$x > 3$," "$x = y + 3$," "$x + y = z$,"

No rules of propositional logic allow us to conclude the truth of the statement

will see how predicate logic can be used to express the meaning of a wide range of statements in mathematics and computer science in ways that permit us to reason and explore relationships between objects.

- “x > 3,”

We can denote the statement “x is greater than 3” by P (x), where P denotes the predicate “is greater than 3” and x is the variable.

The statement *P (x)* is also said to be the value of the **propositional function** *P* at *x*. Once a value has been assigned to the variable x, the statement P (x) becomes a proposition and has a truth value.

Example

Let P (x) denote the statement “x > 3.” What are the truth values of P (4) and P (2)?

Solution

P (4) by setting x = 4 in the statement “x > 3.” Hence, P (4) is true

P (2) by setting x = 2 in the statement “x > 3.” Hence, P (2) is false

We can also have **propositional functions** that involve more than one variable.

Example

Let Q(x, y) denote the statement "$x = y + 3$." What are the truth values of the propositions Q(1, 2) and Q(3, 0)?

Solution

Q(1, 2), set $x = 1$ and $y = 2$ in the statement Q(x, y). Hence, Q(1, 2) is the statement "$1 = 2 + 3$," which is false.

Q(3, 0) is the proposition "$3 = 0 + 3$," which is true.

Example

Let R(x, y, z) denote the statement "$x + y = z$." What are the truth values of the propositions R(1, 2, 3) and R(0, 0, 1) ?

Lecture 2

Solution

The proposition R(1, 2, 3) is obtained by setting $x = 1$, $y = 2$, and $z = 3$ in the statement R(x, y, z). We see that R(1, 2, 3) is the statement "$1 + 2 = 3$," which is true.

R(0, 0, 1), which is the statement "$0 + 0 = 1$," is false.

A statement of the form $P(x1, x2, \ldots, xn)$ is the value of the **propositional function** P at the n-tuple $(x1, x2, \ldots, xn)$, and P is also called an n-**place predicate** or a n-**ary predicate**.

Quantifiers

Quantification expresses the extent to which a predicate is true over a range of elements

We need quantifiers to express the meaning of English words including all and some

The universal quantifier

The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the **universal quantifier.** We read $\forall x P(x)$ as "for all $xP(x)$" or "for every x $P(x)$." An element for which $P(x)$ is false is called a **counterexample** of $\forall x P(x)$

Lecture 2

Example

Let *P* (*x*) be the statement "$x + 1 > x$." What is the truth value of the quantification $\forall x P(x)$, where the domain consists of all real numbers?

$\forall x P(x)$ is true.

Example

Let *P* (*x*) be the statement "$x > 0$." What is the truth value of the quantification $\forall x P(x)$,

- where the domain consists of all integers numbers?
  $\forall x P(x)$ is false.
- where the domain consists of all positive integers numbers?
  $\forall x P(x)$ is true.

Example

Let *P* (*x*) be the statement "x is even." What is the truth value of the quantification $\forall x P(x)$, where the domain consists of all real numbers?

$\forall x P(x)$ is false.

Example

What is the truth value of $\forall x P(x)$, where P (x) is the statement "$x^2 < 10$" and the domain consists of the positive integers not exceeding 4?

Solution

The statement $\forall x P(x)$ is the same as the conjunction

$$P(1) \wedge P(2) \wedge P(3) \wedge P(4),$$

because the domain consists of the integers 1, 2, 3, and 4. Because P (4), which is the statement "$4^2 < 10$," is false, it follows that $\forall x P(x)$ is false.

Example

What does the statement $\forall x N(x)$ mean if N(x) is "Computer x is connected to the network" and the domain consists of all computers on campus?

Lecture 2

Solution

The statement $\forall x N(x)$ means that for every computer x on campus, that computer x is connected to the network. This statement can be expressed in English as

"Every computer on campus is connected to the network."

Example

What is the truth value of $\forall x(x^2 \geq x)$ if the domain consists of all real numbers? What is the truth value of this statement if the domain consists of all integers?

Solution

- $\forall x(x^2 \geq x)$ where the domain consists of all real numbers, is false. For example, $(\frac{1}{2})^2 \not\geq \frac{1}{2}$.
- $\forall x(x^2 \geq x)$ is true, because there are no integers x with $0 < x < 1$.

The Existential quantifier

We use the notation ∃xP (x) for the existential quantification of P (x). Here ∃ is called the existential quantifier

∃xP (x) is read as

"There is an x such that P (x),"

"There is at least one x such that P (x),"

or

"For some xP (x)."

Example

Let *P (x)* be the statement "*x > 3*." What is the truth value of the quantification ∃xP (x), where the domain consists of all real numbers?

∃xP (x), is true.

- Observe that the statement ∃xP (x) is false if and only if there is no element x in the domain for which P (x) is true.

Example

Let *P (x)* be the statement "*x > 0*." What is the truth value of the quantification ∃xP (x)

- where the domain consists of all integers numbers?
  ∃xP (x) is true.
- where the domain consists of all negative integers numbers?
  ∀xP (x) is false.

Lecture 2

The uniqueness quantifier

the **uniqueness quantifier**, denoted by ∃! or ∃1.The notation ∃! $xP(x)$ [or ∃1$xP(x)$] states "There exists a unique x such that $P(x)$ is true."

"there is exactly one" and "there is one and only one.") For instance, ∃!x(x − 1 = 0), where the domain is the set of real numbers,

TABLE 1 Quantifiers.

Statement	When True?	When False?
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Quantifiers with Restricted Domains

Example

What do the statements $\forall x < 0\ (x^2 > 0)$, $\forall y = 0\ (y^3 = 0)$, and $\exists z > 0\ (z^2 = 2)$ mean, where the domain in each case consists of the real numbers?

Solution

The statement $\forall x < 0\ (x^2 > 0)$ is the same as $\forall x\ (x < 0 \rightarrow x^2 > 0)$.

The statement $\forall y = 0\ (y^3 = 0)$, this statement is equivalent to $\forall y(y = 0 \rightarrow y^3 = 0)$

the statement $\exists z > 0\ (z^2 = 2)$, this statement is equivalent to $\exists z(z > 0 \wedge z^2 = 2)$.

Precedence of Quantifiers

Lecture 2

The quantifiers $\forall$ and $\exists$ have higher precedence than all logical operators from propositional calculus.
For example,
$\forall xP(x) \vee Q(x)$ is the disjunction of $\forall xP(x)$ and $Q(x)$. In other words, it means $(\forall xP(x)) \vee Q(x)$ rather than $\forall x(P(x) \vee Q(x))$.

• Logical Equivalences Involving Quantifiers

Statements involving predicates and quantifiers are logically equivalent if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions. $S \equiv T$

Example

Show that $\forall x(P(x) \wedge Q(x))$ and $\forall xP(x) \wedge \forall xQ(x)$ are logically equivalent

we must show that they always take the same truth value, no matter what the predicates P and Q are, and no matter which domain of discourse is used.

Suppose we have particular predicates P and Q, with a common domain.

- First, we show that if $\forall x(P(x) \wedge Q(x))$ is true, then $\forall xP(x) \wedge \forall xQ(x)$ is true.
- Second, we show that if $\forall xP(x) \wedge \forall xQ(x)$ is true, then $\forall x(P(x) \wedge Q(x))$ is true.

So, suppose that $\forall x(P(x) \land Q(x))$ is true. This means that if a is in the domain, then $P(a) \land Q(a)$ is true. Hence, $P(a)$ is true and $Q(a)$ is true. Because $P(a)$ is true and $Q(a)$ is true for every element in the domain, we can conclude that $\forall x P(x)$ and $\forall x Q(x)$ are both true. This means that $\forall x P(x) \land \forall x Q(x)$ is true.

Next, suppose that $\forall x P(x) \land \forall x Q(x)$ is true. It follows that $\forall x P(x)$ is true and $\forall x Q(x)$ is true. Hence, if a is in the domain, then $P(a)$ is true and $Q(a)$ is true . It follows that for all a, $P(a) \land Q(a)$ is true. It follows that $\forall x(P(x) \land Q(x))$ is true. We can now conclude that:

$$\forall x(P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x).$$

Lecture 2

Negating Quantified Expressions

"Every student in your class has taken a course in calculus."

This statement is a universal quantification, namely $\forall x P(x)$

where P (x) is the statement "x has taken a course in calculus" and the domain consists of the students in your class

The negation of this statement is
"It is not the case that every student in your class has taken a course in calculus."

This is equivalent to

"There is a student in your class who has not taken a course in calculus."

$\exists x \neg P(x)$.

This example illustrates the following logical equivalence:

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

There is a student in this class who has taken a course in calculus."
This is the existential quantification $\exists x\, Q(x)$,

The negation of this statement is the proposition

"It is not the case that there is a student in this class who has taken a course in calculus."

This is equivalent to

"Every student in this class has not taken calculus," $\forall x\, \neg Q(x)$.

$$\neg \exists x\, Q(x) \equiv \forall x\, \neg Q(x).$$

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Lecture 2

Example

What are the negations of the statements $\forall x\ (x^2 > x)$ and $\exists x(x^2 = 2)$?

Solution

The negation of $\forall x(x^2 > x)$ is the statement $\neg\forall x(x^2 > x)$, which is equivalent to $\exists x\neg(x^2 > x)$. This can be rewritten as $\exists x(x^2 \le x)$.

The negation of $\exists x\ (x^2 = 2)$ is the statement $\neg\exists x\ (x^2 = 2)$, which is equivalent to $\forall x\ \neg\ (x^2 = 2)$. This can be rewritten as $\forall x(x^2 \neq 2)$.

Example

Show that $\neg\forall x(P(x) \rightarrow Q(x))$ and $\exists x(P(x) \wedge \neg Q(x))$ are logically equivalent.

Lecture 2

$\neg\forall x\ (P(x) \rightarrow Q(x)) \equiv \exists x\ (\neg(P(x) \rightarrow Q(x)))$ (De Morgan's law for universal quantifiers)

$\equiv \exists x\ (P(x) \wedge \neg Q(x))$ (By the fifth logical equivalence in Table 7)

Translating from English into Logical Expressions

Example

Express the statements "Some student in this class has visited Mexico" and "Every student in this class has visited either Canada or Mexico" using predicates and quantifiers.

Solution

We introduce $M(x)$, which is the statement "x has visited Mexico." and

$S(x)$ to represent "x is a student in this class."

the domain for the variable x consists of all people.

- Our solution becomes $\exists x\,(S(x) \wedge M(x))$

Let $C(x)$, which is the statement "x has visited Canada ."

$$\forall x(S(x) \rightarrow (C(x) \vee M(x)))$$

Lecture 2

• Nested Quantifiers

Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.

- We will see how to use nested quantifiers to express mathematical statements such as

  "The sum of two positive integers is always positive."

- We will show how nested quantifiers can be used to translate English sentences such as

  "Everyone has exactly one best friend"

Into logical statements.

Lecture 2

Understanding Statements Involving Nested Quantifiers

we need to unravel what the quantifiers and predicates that appear mean.

Example: "Every real number has an inverse" is

$$\forall x\, \exists y\, (x + y = 0)$$

where the domains of x and y are the real numbers.

We can also think of nested propositional functions:
$\forall x\, \exists y\, (x + y = 0)$ can be viewed as $\forall x\, Q(x)$ where $Q(x)$ is, $\exists y\, P(x, y)$ where $P(x, y)$ is $(x + y = 0)$.

Lecture 2

Assume that the domain for the variables x and y consists of all real numbers. The statement

- $\forall x\, \forall y\, (x + y = y + x)$

says that $x + y = y + x$ for all real numbers x and y.

- $\forall x\, \exists y\, (x + y = 0)$

says that for every real number x there is a real number y such that $x + y = 0$.

this states that every real number has an additive inverse

Example

Translate into English the statement

$$\forall x\, \forall y\, ((x > 0) \wedge (y < 0) \rightarrow (xy < 0)),$$

where the domain for both variables consists of all real numbers.

Solution

this statement says that for real numbers x and y, if x is positive and y is negative, then xy is negative.

The Order of Quantifiers

It is important to note that the order of the quantifiers is important, unless all the quantifiers are universal quantifiers or all are existential quantifiers.

Example

Let $P(x, y)$ be the statement "$x + y = y + x$." What are the truth values of the quantifications $\forall x \forall y P(x, y)$ and $\forall y \forall x P(x, y)$ where the domain for all variables consists of all real numbers?

$\forall x \forall y P(x, y)$ and $\forall y \forall x P(x, y)$ have the same meaning, and both are true

Example

Let $Q(x, y)$ denote "$x + y = 0$." What are the truth values of the quantifications $\exists y \forall x Q(x, y)$ and $\forall x \exists y Q(x, y)$, where the domain for all variables consists of all real numbers?

Solution

$$\exists y \forall x Q(x, y)$$

denotes the proposition:

"There is a real number y such that for every real number x, Q(x, y)."

Because

there is no real number y such that $x + y = 0$ for all real numbers x, the statement $\exists y \forall x Q(x, y)$ is false

The quantification

$$\forall x \, \exists y \, Q(x, y)$$

denotes the proposition

“For every real number x there is a real number y such that Q(x, y).”

Given a real number x, there is a real number y such that $x + y = 0$; namely, $y = -x$. Hence, the statement $\forall x \exists y Q(x, y)$ is true

TABLE 1 Quantifications of Two Variables.

Statement	*When True?*	*When False?*
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Lecture 2

Define $P(x,y) : x \cdot y = 0$, where the domain for all variables consists of all real numbers. What is the truth value of the following:

$\forall x \forall y\ P(x,y)$ False

$\forall x\ \exists y\ P(x,y)$ True

$\exists x\ \forall y\ P(x,y)$ True

$\exists x\ \exists y\ P(x,y)$ True

Quiz (1)

Define $P(x,y) : x / y = 1$, where the domain for all variables consists of all real numbers. What is the truth value of the following:

$\forall x \forall y\ P(x,y)$ ……………..……

$\forall x\ \exists y\ P(x,y)$ ………………..

$\exists x\ \forall y\ P(x,y)$ ………………..

$\exists x\ \exists y\ P(x,y)$ ………………

Lecture 2

Translating Mathematical Statements into Statements Involving Nested Quantifiers

Example

Translate the statement "The sum of two positive integers is always positive" into a logical expression

we can express this statement as

$$\forall x\, \forall y\, ((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$$

where the domain for both variables consists of all integers.

Note that we could also translate this using the positive integers as the domain. We can express this as

$$\forall x\, \forall y\, (x + y > 0).$$

Example

Translate the statement "Every real number except zero has a multiplicative inverse."

This can be rewritten as

$$\forall x\, ((x \neq 0) \rightarrow \exists y\, (xy = 1)).$$

Lecture 2

Thinking of Nested Quantification

Nested Loops

- To see if $\forall x \forall y\ p(x,y)$ is true, loop through the values of x :
- At each step, loop through the values for y.
- If for some pair of x and y, $P(x,y)$ is false, then $\forall x \forall y\ p(x,y)$ is false and both the outer and inner loop terminate.
- $\forall x \forall y\ p(x,y)$ is true if the outer loop ends after stepping through each x

- To see if $\forall x \exists y\ P(x,y)$ is true, loop through the values of x:
- At each step, loop through the values for y.
- The inner loop ends when a pair x and y is found such that $P(x, y)$ is true.
- If no y is found such that $P(x, y)$ is true the outer loop terminates as $\forall x \exists y\ P(x,y)$ has been shown to be false.
- $\forall x \exists y\ P(x,y)$ is true if the outer loop ends after stepping through each x.

If the domains of the variables are infinite, then this process can not actually be carried out.

Lecture 2

Negating Nested Quantifiers

Example

Express the negation of the statement $\forall x\ \exists y\ (xy = 1)$

Solution

applying De Morgan's laws for quantifiers, we can move the negation in $\neg\forall x\exists y(xy = 1)$ inside all the quantifiers. We find that

$$\neg\forall x\ \exists y\ (xy = 1) \equiv \exists x\ \neg\exists y\ (xy = 1),$$
$$\equiv \exists x\ \forall y\ \neg(xy = 1).$$
$$\equiv \exists x\ \forall y\ (xy \neq 1).$$

we conclude that our negated
statement can be expressed as ∃ $\exists x\ \forall y\ (xy \neq 1)$.

• Rules of Inference

Valid Arguments

By an argument, we mean a sequence of statements that end with a conclusion. By valid, we mean that the conclusion, or final statement of the argument, must follow from the truth of the preceding statements, or premises, of the argument

- We will show how to construct valid arguments in two stages;
- first for propositional logic and then for predicate logic. The rules of inference are the essential building block in the construction of valid arguments.

- Propositional Logic
  - Inference Rules
- Predicate Logic
  - Inference rules for propositional logic plus additional inference rules to handle variables and quantifiers.

Lecture 2

Valid Arguments in Propositional Logic

"If you have a current password, then you can log onto the network."

"You have a current password."

Therefore,

"You can log onto the network."

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

We know that when p and q are propositional variables, the statement $((p \rightarrow q) \wedge p) \rightarrow q$ is a tautology. In particular, when both $p \rightarrow q$ and p are true, we know that q must also be true. We say this form of argument is **valid** because whenever all its premises (all statements in the argument other than the final one, the conclusion) are true, the conclusion must also be true.

An **argument** in propositional logic is a sequence of propositions. All but the final proposition in the argument are called premises and the final proposition is called the conclusion. An argument is valid if the truth of all its premises implies that the conclusion is true.

The argument is valid if the premises imply the conclusion

From the definition of a valid argument form we see that the argument form with premises $(p1, p2, \dots, pn)$ and conclusion q is valid, when $(p1 \land p2 \land \cdots \land pn) \rightarrow q$ is a **tautology**

Lecture 2

Rules of Inference for Propositional Logic: Modus Ponens

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

Corresponding Tautology:

$(p \wedge (p \rightarrow q)) \rightarrow q$

Example:

Let p be "It is snowing."
Let q be "I will study discrete math."

"If it is snowing, then I will study discrete math."
"It is snowing."

"Therefore , I will study discrete math."

Lecture 2

Modus Tollens

$$\begin{array}{l} p \to q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

Corresponding Tautology:

$(\neg q \wedge (p \to q)) \to \neg p$

Example:

Let p be "it is snowing."

Let q be "I will study discrete math."

"If it is snowing, then I will study discrete math."

"I will not study discrete math."

"Therefore , it is not snowing."

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Hypothetical Syllogism

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

Corresponding Tautology:
$((p \to q) \land (q \to r)) \to (p \to r)$

Example:
Let p be "it snows."
Let q be "I will study discrete math."
Let r be "I will get an A."

"If it snows, then I will study discrete math."
"If I study discrete math, I will get an A."

"Therefore , If it snows, I will get an A."

Disjunctive Syllogism

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

Corresponding Tautology:
$(\neg p \wedge (p \vee q)) \rightarrow q$

Example:
Let p be "I will study discrete math."
Let q be "I will study English literature."

"I will study discrete math or I will study English literature."
"I will not study discrete math."

"Therefore , I will study English literature."

Lecture 2

Addition

$$\frac{p}{\therefore p \vee q}$$

Corresponding Tautology:

$p \rightarrow (p \vee q)$

Example:

Let p be "I will study discrete math."

Let q be "I will visit Las Vegas."

"I will study discrete math."

"Therefore, I will study discrete math or I will visit Las Vegas."

Lecture 2

Simplification

$$\frac{p \wedge q}{\therefore q}$$

Corresponding Tautology:

$(p \wedge q) \rightarrow p$

Example:

Let *p* be "I will study discrete math."

Let *q* be "I will study English literature."

"I will study discrete math and English literature"

"Therefore, I will study discrete math."

Conjunction

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

Corresponding Tautology:

$((p) \wedge (q)) \rightarrow (p \wedge q)$

Example:

Let p be "I will study discrete math."

Let q be "I will study English literature."

"I will study discrete math."

"I will study English literature."

"Therefore, I will study discrete math and I will study English literature."

Lecture 2

Resolution

$$\begin{array}{l} \neg p \vee r \\ p \vee q \\ \hline \therefore q \vee r \end{array}$$

Corresponding Tautology:

$((\neg p \vee r) \wedge (p \vee q)) \rightarrow (q \vee r)$

Example:

Let *p* be "I will study discrete math."
Let *r* be "I will study English literature."
Let q be "I will study databases."

"I will not study discrete math or I will study English literature."
"I will study discrete math or I will study databases."

"Therefore, I will study databases or I will study English literature."

Lecture 2

Using the Rules of Inference to Build Valid Arguments

- A *valid argument* is a sequence of statements. Each statement is either a premise or follows from previous statements by rules of inference. The last statement is called conclusion.
- A valid argument takes the following form:

$$S_1$$
$$S_2$$
$$.$$
$$.$$
$$.$$
$$S_n$$

$$C$$

Lecture 2

Example

From the single proposition

$$p \wedge (p \to q)$$

Show that q is a conclusion.

Solution:

Step	**Reason**
1. $p \wedge (p \to q)$	Premise
2. p	Simplification using (1)
3. $p \to q$	Simplification using (1)
4. q	Modus Ponens using (2) and (3)

Example

Show that the premises
With these hypotheses:

"It is not sunny this afternoon and it is colder than yesterday."
"We will go swimming only if it is sunny."
"If we do not go swimming, then we will take a canoe trip."
"If we take a canoe trip, then we will be home by sunset."

Using the inference rules, construct a valid argument for the conclusion:

"We will be home by sunset."

Solution

1. Choose propositional variables:

p : "It is sunny this afternoon." r : "We will go swimming."
t : "We will be home by sunset." q : "It is colder than yesterday."
s : "We will take a canoe trip."

2. Translation into propositional logic:

Hypotheses: $\neg p \wedge q,\ r \rightarrow p,\ \neg r \rightarrow s,\ s \rightarrow t$
Conclusion: t

Lecture 2

3. Construct the Valid Argument

Step	**Reason**
1. $\neg p \wedge q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \rightarrow p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \rightarrow s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \rightarrow t$	Premise
8. t	Modus ponens using (6) and (7)

Lecture 2

- Valid arguments for quantified statements are a sequence of statements. Each statement is either a premise or follows from previous statements by rules of inference which include:
  - Rules of Inference for Propositional Logic
  - Rules of Inference for Quantified Statements

Lecture 2

Rules of Inference for Quantified Statements

TABLE 2 Rules of Inference for Quantified Statements.

Rule of Inference	*Name*
$\forall x P(x)$ $\therefore P(c)$	Universal instantiation
$P(c)$ for an arbitrary c $\therefore \forall x P(x)$	Universal generalization
$\exists x P(x)$ $\therefore P(c)$ for some element c	Existential instantiation
$P(c)$ for some element c $\therefore \exists x P(x)$	Existential generalization

Lecture 2

Example 1: Using the rules of inference, construct a valid argument to show that

"John Smith has two legs"

is a consequence of the premises:

"Every man has two legs." "John Smith is a man."

Solution: Let $M(x)$ denote "x is a man" and $L(x)$ " x has two legs" and let John Smith be a member of the domain.

Valid Argument:

Step	Reason
1. $\forall x(M(x) \rightarrow L(x))$	Premise
2. $M(J) \rightarrow L(J)$	UI from (1)
3. $M(J)$	Premise
4. $L(J)$	Modus Ponens using (2) and (3)

Lecture 2

Example 2:

Use the rules of inference to construct a valid argument showing that the conclusion

"Someone who passed the first exam has not read the book."

follows from the premises

"A student in this class has not read the book."

"Everyone in this class passed the first exam."

Solution:

Let $C(x)$ denote "x is in this class," $B(x)$ denote " x has read the book," and $P(x)$ denote "x passed the first exam."

First we translate the premises and conclusion into symbolic form.

$$\begin{array}{l} \exists x(C(x) \land \neg B(x)) \\ \forall x(C(x) \rightarrow P(x)) \\ \hline \therefore\ \exists x(P(x) \land \neg B(x)) \end{array}$$

Lecture 2

Valid Argument:

Step	**Reason**
1. $\exists x(C(x) \land \neg B(x))$	Premise
2. $C(a) \land \neg B(a)$	EI from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \to P(x))$	Premise
5. $C(a) \to P(a)$	UI from (4)
6. $P(a)$	MP from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \land \neg B(a)$	Conj from (6) and (7)
9. $\exists x(P(x) \land \neg B(x))$	EG from (8)