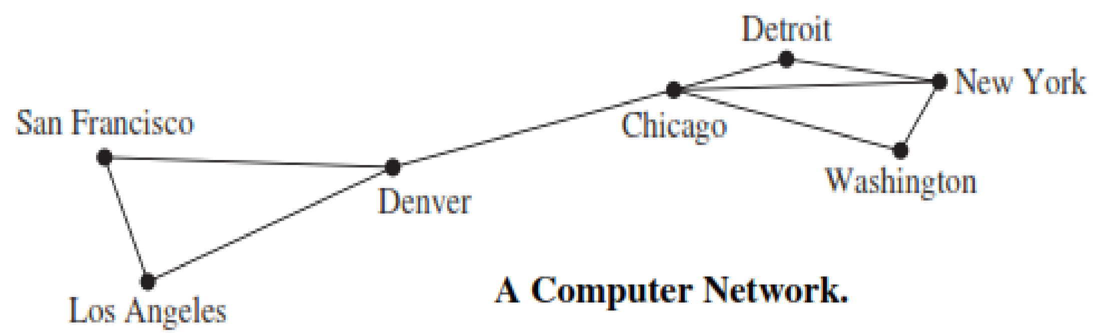
CH.10 Graphs

- ❑ 10.1 Graphs and Graph Models
- ❑ 10.2 Graph Terminology and Special Types of Graphs
- ❑ 10.3 Representing Graphs and Graph Isomorphism
- ❑ 10.4 Connectivity

Lecture 8

Simple Graph

- A ***simple graph*** consists of
  - a nonempty set of *vertices* called V
  - a set of *edges* (unordered pairs of distinct elements of V) called E
- Notation: $G = (V,E)$



A Computer Network.

- This simple graph represents a network.
- The network is made up of computers and telephone links between computers

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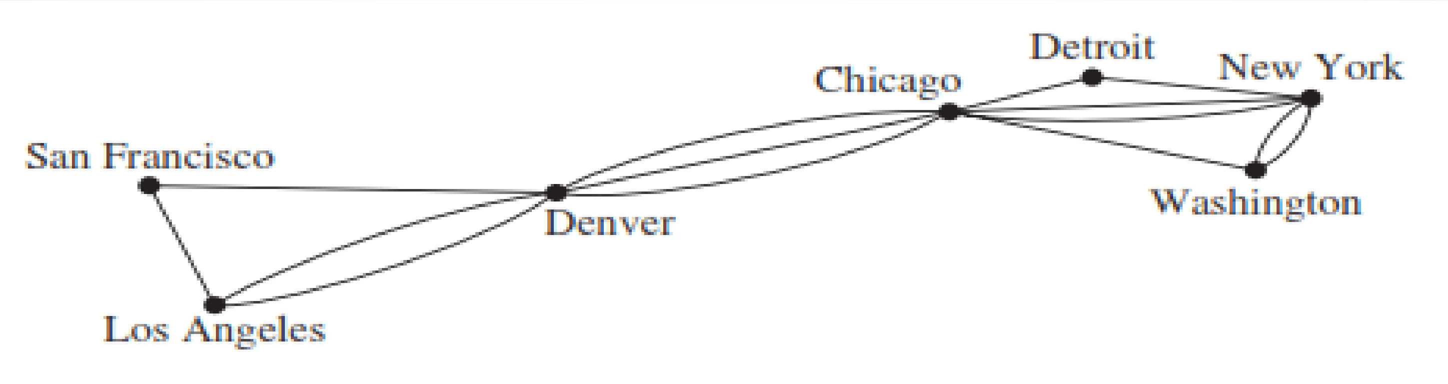
Multigraph

A **multigraph** can have multiple edges (two or more edges connecting the same pair of vertices).



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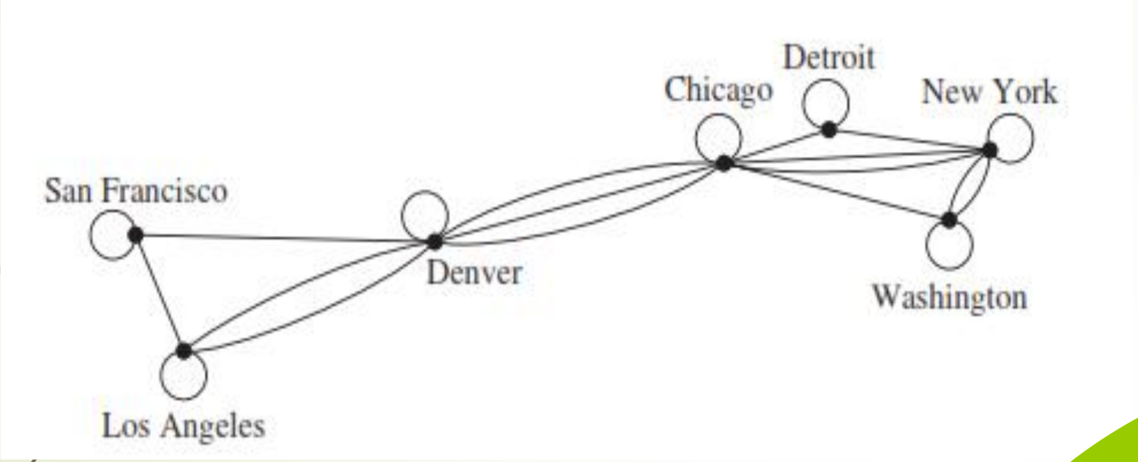
- There can be multiple telephone lines between two computers in the network.

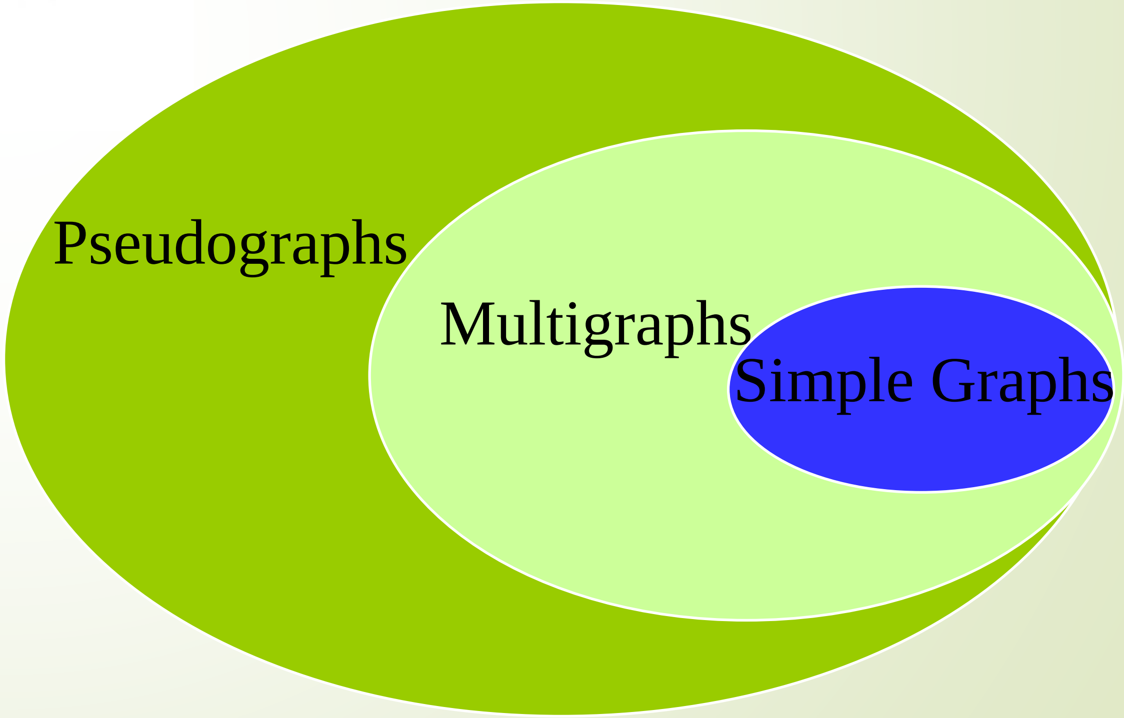
Pseudograph

A Pseudograph can have multiple edges and loops (an edge connecting a vertex to itself).



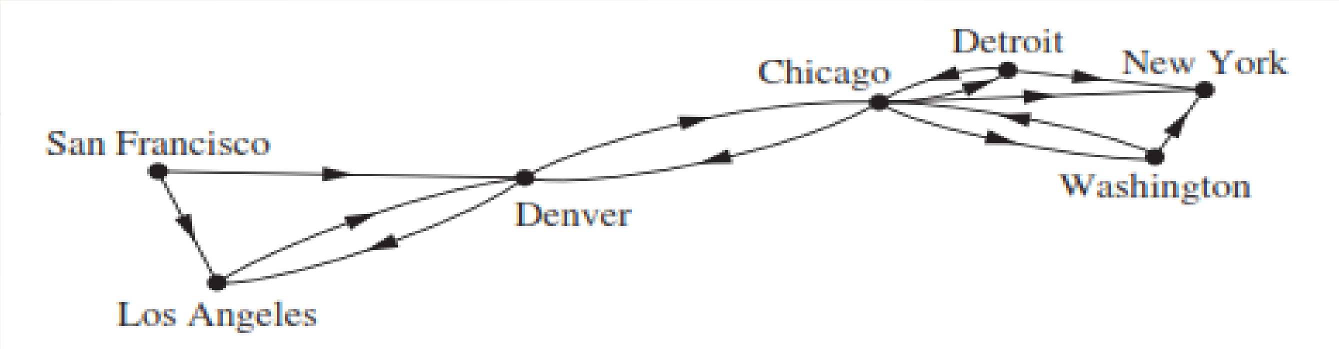
There can be telephone lines in the network from a computer to itself.

Types of Undirected Graphs



Directed Graph

The edges are ordered pairs of (not necessarily distinct) vertices.



Some telephone lines in the network may operate in only one direction. Those that operate in two directions are represented by pairs of edges in opposite directions.

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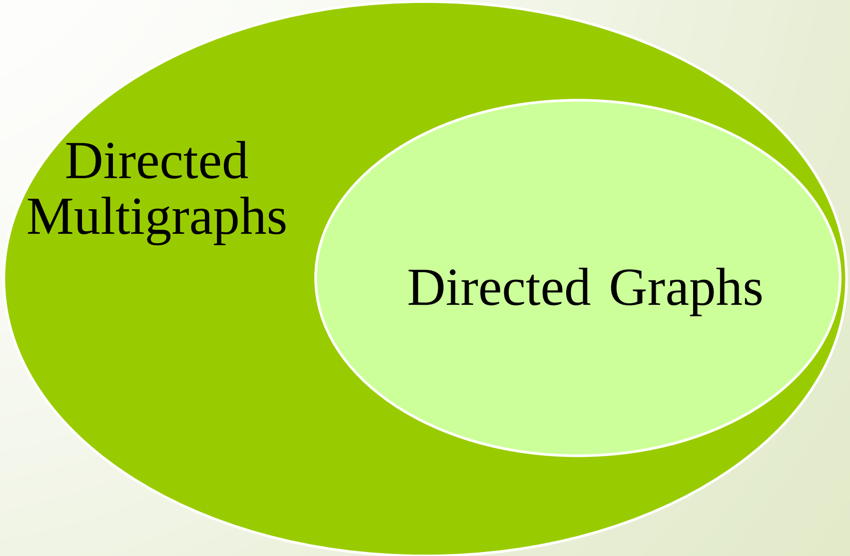
Types of Directed Graphs



Summary

Type	Edges	Loops	Multiple Edges
Simple Graph	Undirected	NO	NO
Multigraph	Undirected	NO	YES
Pseudograph	Undirected	YES	YES
Simple Directed Graph	Directed	NO	NO
Directed multigraph	Directed	YES	YES
Mixed graph	Directed and undirected	YES	YES

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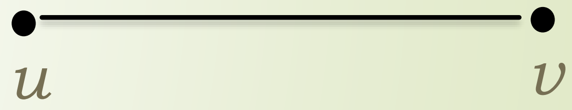
❑ 10.2 Graph Terminology and Special Types of Graphs

Basic Terminology

❑ Adjacent Vertices in Undirected Graphs

- Two vertices, *u* and *v* in an undirected graph *G* are called ***adjacent*** (or neighbors) in *G*, if {*u*,*v*} is an edge of *G*.
- An edge *e* connecting *u* and *v* is called ***incident*** with vertices *u* and *v*, or is said to connect *u* and *v*.
- The vertices *u* and *v* are called ***endpoints*** of edge {*u*,*v*}.



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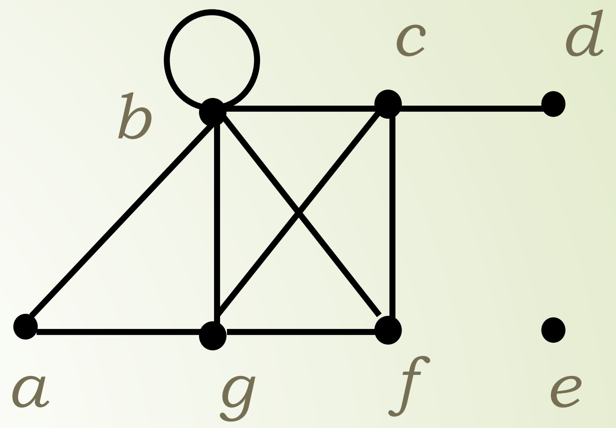
❑ Degree of a Vertex

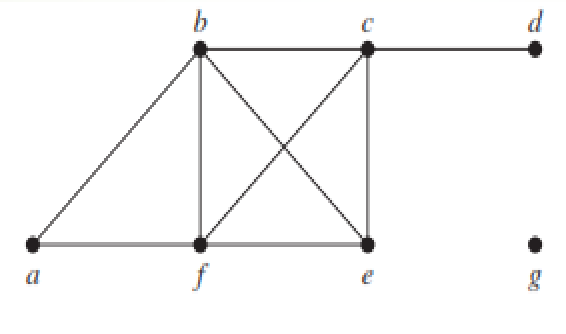
- The ***degree of a vertex*** in an undirected graph is the number of edges incident with it
  - except that a loop at a vertex contributes twice to the degree of that vertex
- The degree of a vertex *v* is denoted by deg(*v*).

Examples

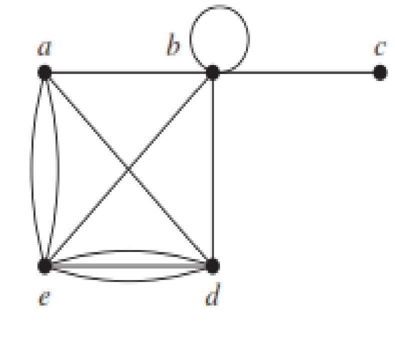
Find the degrees of all the vertices:

deg(a) = 2, deg(b) = 6, deg(c) = 4, deg(d) = 1,
deg(e) = 0, deg(f) = 3, deg(g) = 4





G



H

Solution: In G, $\deg(a) = 2$, $\deg(b) = \deg(c) = \deg(f) = 4$, $\deg(d) = 1$, $\deg(e) = 3$, and $\deg(g) = 0$. The neighborhoods of these vertices are $N(a) = \{b, f\}$, $N(b) = \{a, c, e, f\}$, $N(c) = \{b, d, e, f\}$, $N(d) = \{c\}$, $N(e) = \{b, c, f\}$, $N(f) = \{a, b, c, e\}$, and $N(g) = \emptyset$. In H, $\deg(a) = 4$, $\deg(b) = \deg(e) = 6$, $\deg(c) = 1$, and $\deg(d) = 5$. The neighborhoods of these vertices are $N(a) = \{b, d, e\}$, $N(b) = \{a, b, c, d, e\}$, $N(c) = \{b\}$, $N(d) = \{a, b, e\}$, and $N(e) = \{a, b, d\}$. ◀

THEOREM 1 **THE HANDSHAKING THEOREM** Let $G = (V, E)$ be an undirected graph with m edges. Then

$$2m = \sum_{v \in V} \deg(v).$$

(Note that this applies even if multiple edges and loops are present.)

Example

How many edges are there in a graph with 10 vertices each of degree six?

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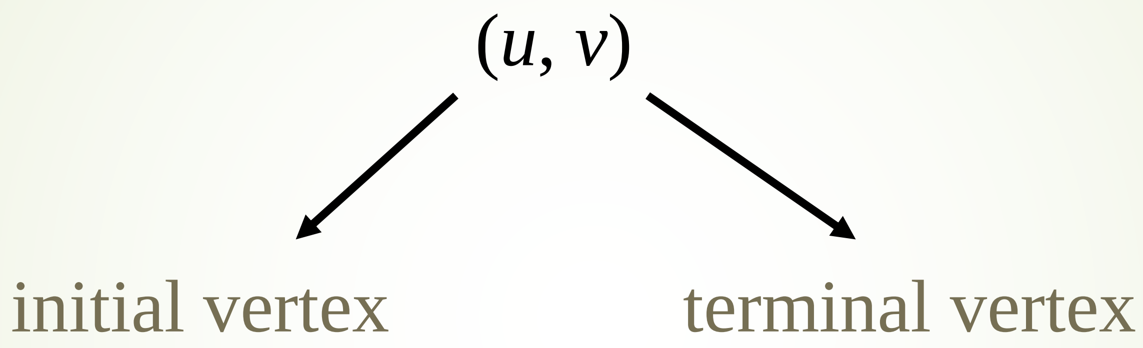
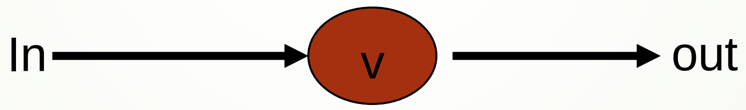
Solution: Because the sum of the degrees of the vertices is $6 \cdot 10 = 60$, it follows that $2m = 60$
where m is the number of edges. Therefore, $m = 30$.

❑ Adjacent Vertices in Directed Graphs

- When (u,v) is an edge of a directed graph G, u is said to be *adjacent to v* and v is said to be *adjacent from u.*



❑ Degree of a Vertex



In-degree of a vertex v

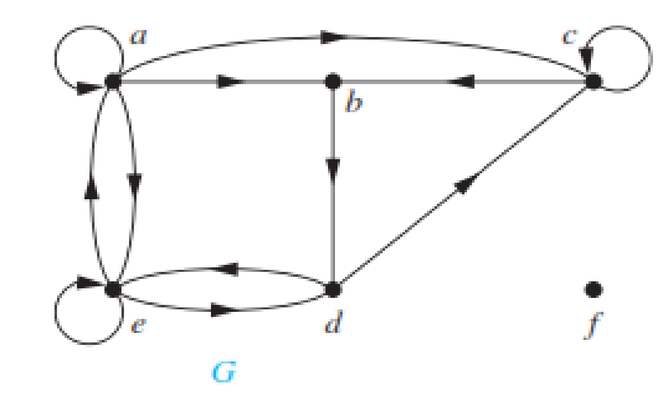
- The number of vertices *adjacent to v* (the number of edges with v as their terminal vertex
- Denoted by $\deg^{-}(v)$

- *Out-degree* of a vertex *v*
  - The number of vertices *adjacent from v* (the number of edges with *v* as their initial vertex)
  - Denoted by $\deg^+(v)$

- A loop at a vertex contributes 1 to both the in-degree and out-degree.

Examples

Find the in-degrees and out-degrees of this digraph.



In-degrees: $\deg^-(a) = 2$, $\deg^-(b) = 2$, $\deg^-(c) = 3$, $\deg^-(d) = 2$, $\deg^-(e) = 3$, $\deg^-(f) = 0$

Out-degrees: $\deg^+(a) = 4$, $\deg^+(b) = 1$, $\deg^+(c) = 2$, $\deg^+(d) = 2$, $\deg^+(e) = 3$, $\deg^+(f) = 0$

THEOREM 3 Let $G = (V, E)$ be a graph with directed edges. Then

$$\sum_{v \in V} \deg^{-}(v) = \sum_{v \in V} \deg^{+}(v) = |E|.$$

The sum of the in-degrees of all vertices in a digraph =

= the sum of the out-degrees

= the number of edges

Some Special Simple Graphs

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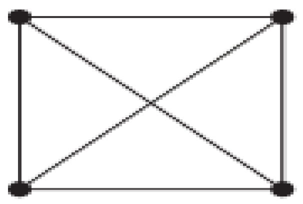
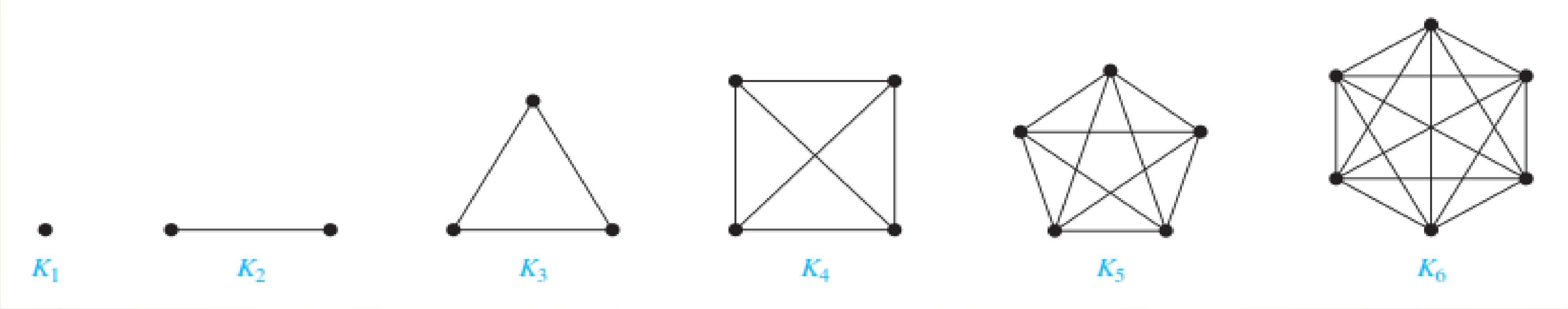
1. Complete Graph



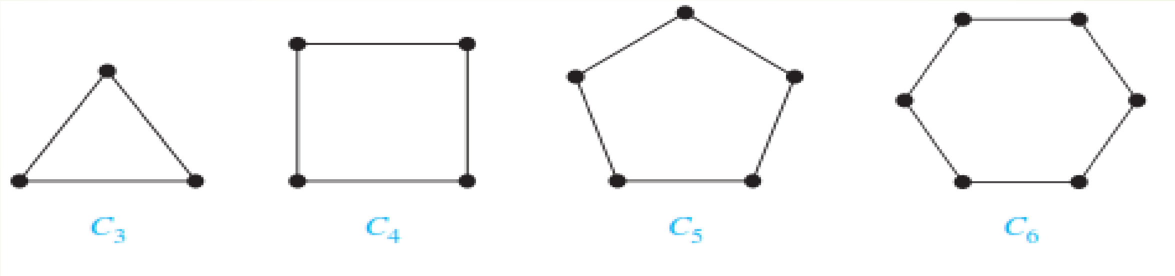
- The *complete graph* on n vertices (K_n) is the simple graph that contains exactly one edge between each pair of distinct vertices.



The figures above represent the complete graphs, K_n, for n = 1, 2, 3, 4, 5, and 6.

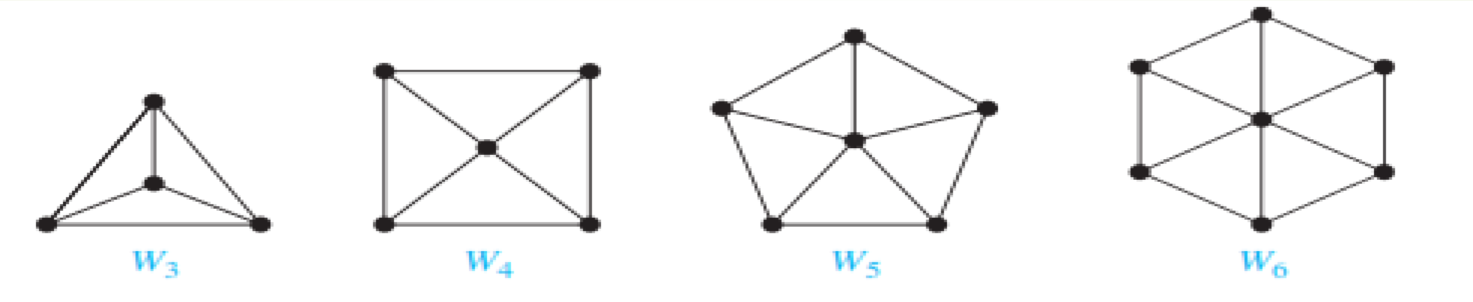
2. Cycle

The *cycle* $\mathbf{C_n}$ ($n \geq 3$), consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1,v_2\}, \{v_2,v_3\}, \ldots, \{v_{n-1},v_n\}$, and $\{v_n,v_1\}$.



3. Wheel

When a new vertex is added to a cycle C_n and this new vertex is connected to each of the n vertices in C_n, we obtain a *wheel* W_n.



4. Subgraph

A *subgraph* of a graph $G = (V,E)$ is a graph $H = (W,F)$ where $W \subseteq V$ and $F \subseteq E$.

Is C_5 a subgraph of K_5?

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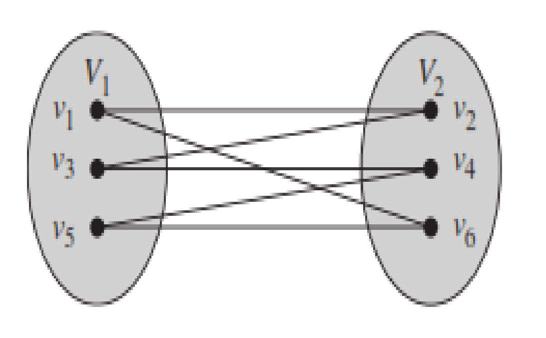
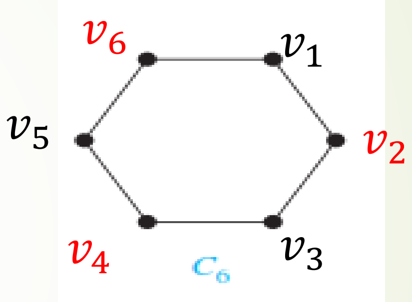
4. Bipartite Graphs

A simple graph G is called *bipartite* if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). When this condition holds, we call the pair (V_1, V_2) a *bipartition* of the vertex set V of G.





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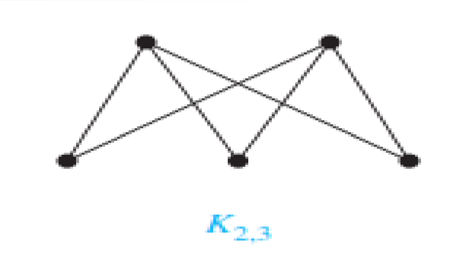
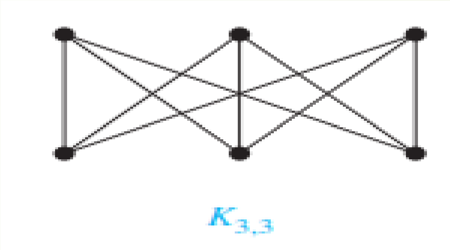
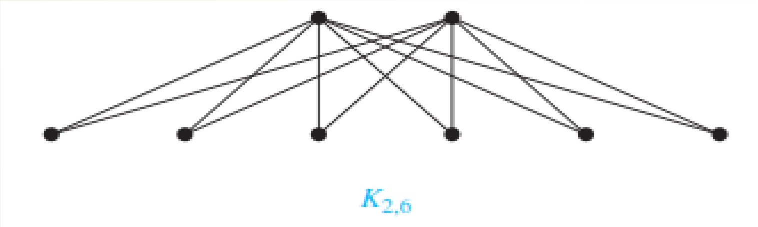
C_6 is bipartite, as shown in Figure 7, because its vertex set can be partitioned into the two sets $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$, and every edge of C_6 connects a vertex in V_1 and a vertex in V_2.

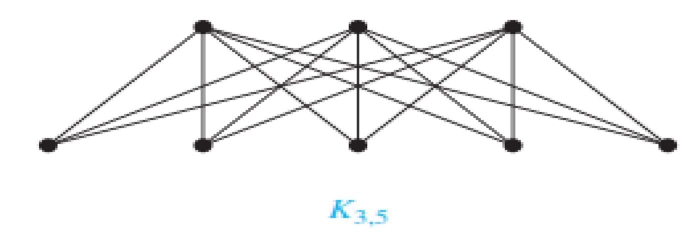
5. Complete Bipartite Graphs

A **complete bipartite graph K**m,n :is a graph that has its vertex set partitioned into two subsets of m and n vertices, respectively with an edge between two vertices if and only if one vertex is in the first subset and the other vertex is in the second subset.



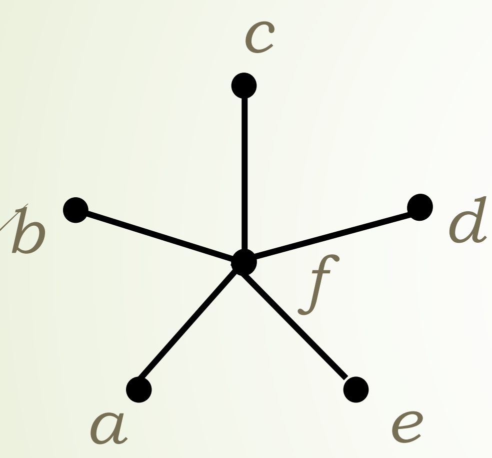




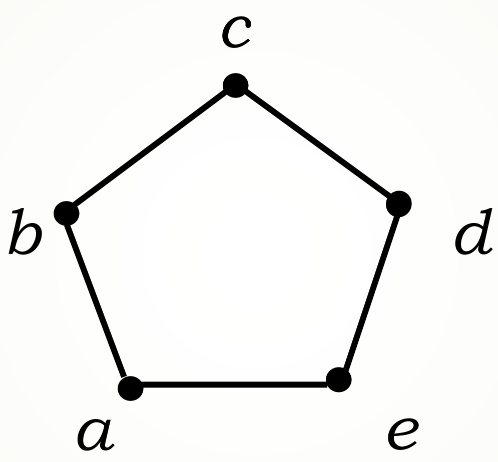


5. Union

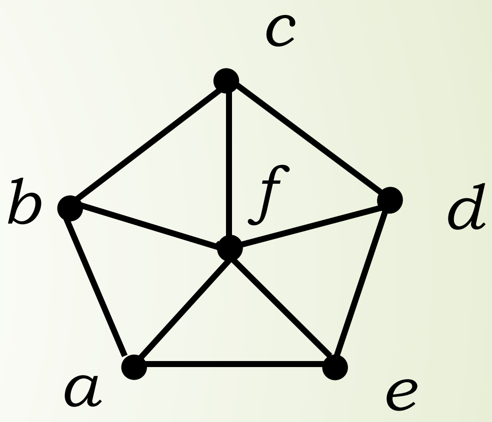
The *union* of 2 simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$. The union is denoted by $G_1 \cup G_2$.



S_5



C_5



W_5

$$S_5 \cup C_5 = W_5$$

❑ 10.3 Representing Graphs and Graph Isomorphism

Representing Graphs

1. Adjacency Matrix

A simple graph $G = (V,E)$ with n vertices can be represented by its *adjacency matrix*, A, where the entry a_{ij} in row i and column j is:

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge in } G \\ 0 & \text{otherwise} \end{cases}$$

Example

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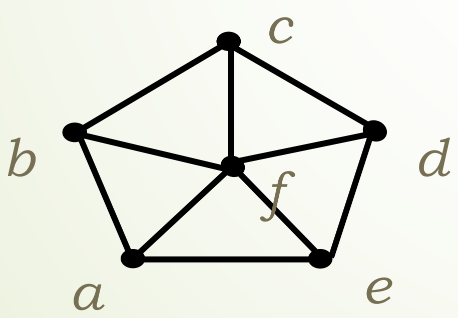


W_5

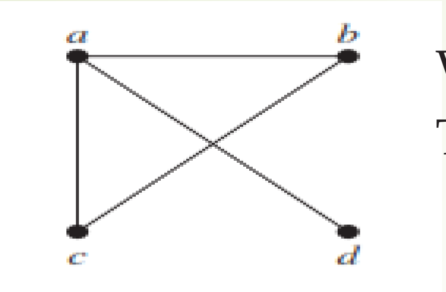
$\{v_1, v_2\}$ → row, column

From (columns) / To (rows)

	a	b	c	d	e	f
a	0	1	0	0	1	1
b	1	0	1	0	0	1
c	0	1	0	1	0	1
d	0	0	1	0	1	1
e	1	0	0	1	0	1
f	1	1	1	1	1	0

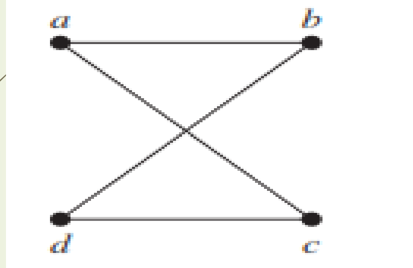
Example

- Use an adjacency matrix to represent the graph



We order the vertices as a, b, c, d.
The matrix representing this graph is

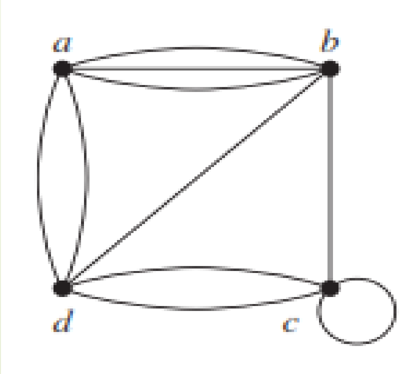
$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$



with respect to the ordering of vertices a, b, c, d.

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

- Use an adjacency matrix to represent the pseudograph



The adjacency matrix using the ordering of vertices a, b, c, d is

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}.$$

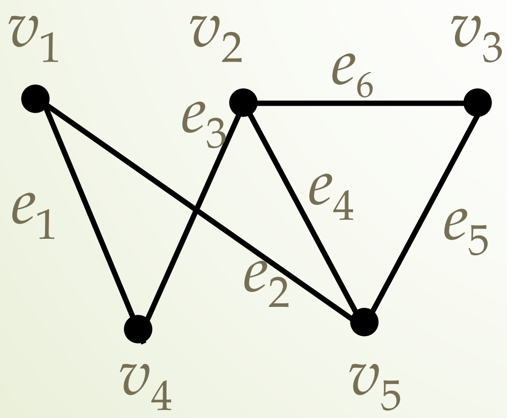
2. Incidence Matrix

Let $G = (V,E)$ be an undirected graph. Suppose $v_1,v_2,v_3,\ldots,v_n$ are the vertices and $e_1,e_2,e_3,\ldots,e_m$ are the edges of G. The *incidence matrix* w.r.t. this ordering of V and E is the $n\times m$ matrix $M = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if edge } e_j \text{ is incident with } v_i \\ 0 & \text{otherwise} \end{cases}$$

Example

Represent the graph shown with an incidence matrix.



edges →

	e_1	e_2	e_3	e_4	e_5	e_6
v_1	1	1	0	0	0	0
v_2	0	0	1	1	0	1
v_3	0	0	0	0	1	1
v_4	1	0	1	0	0	0
v_5	0	1	0	1	1	0

vertices ↑

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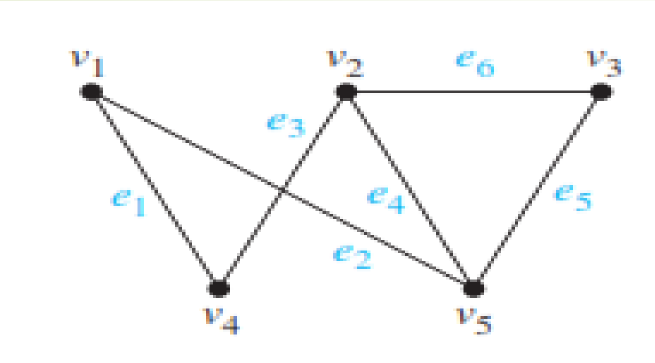
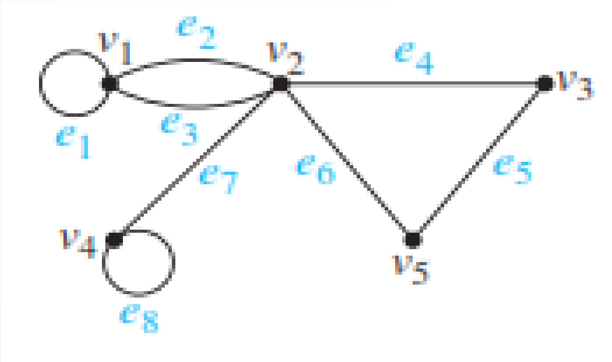
Example

Represent the following graph with an incidence matrix.



The incidence matrix is

	e_1	e_2	e_3	e_4	e_5	e_6
v_1	1	1	0	0	0	0
v_2	0	0	1	1	0	1
v_3	0	0	0	0	1	1
v_4	1	0	1	0	0	0
v_5	0	1	0	1	1	0



The incidence matrix for this graph is

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
v_1	1	1	1	0	0	0	0	0
v_2	0	1	1	1	0	1	1	0
v_3	0	0	0	1	1	0	0	0
v_4	0	0	0	0	0	0	1	1
v_5	0	0	0	0	1	1	0	0

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Isomorphism

Two simple graphs are isomorphic if:

- there is a one-to one correspondence between the vertices of the two graphs
- the adjacency relationship is preserved

DEFINITION

The simple graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are *isomorphic* if there is a one-to-one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 iff $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1.

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Example

Show that the graphs G = (V , E) and H = (W, F), displayed in Figure 8, are isomorphic.

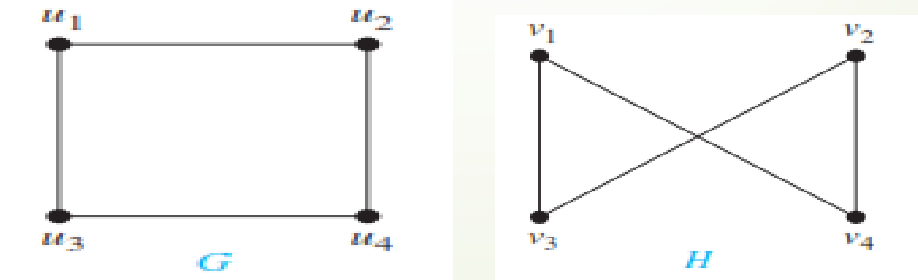


FIGURE 8 The Graphs G and H.

Are G and H isomorphic?

$$f(u_1)=v_1,\ f(u_2)=v_4,\ f(u_3)=v_3,\ f(u_4)=v_2$$

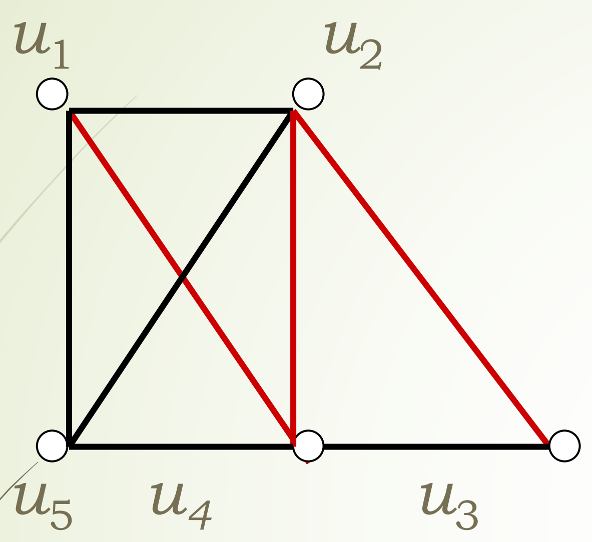
10.4 Connectivity

Paths in Undirected Graphs

- There is a ***path*** from vertex v_0 to vertex v_n if there is a sequence of edges from v_0 to v_n
  - This path is labeled as $v_0, v_1, v_2, \ldots, v_n$ and has a length of n.
- The path is a ***circuit*** if the path begins and ends with the same vertex.
- A path is *simple* if it does not contain the same edge more than once.

- A path or circuit is said to *pass through* the vertices v_0, v_1, v_2, …, v_n or *traverse* the edges e_1, e_2, …, e_n.

Examples



- u_1, u_4, u_2, u_3

Is it simple?

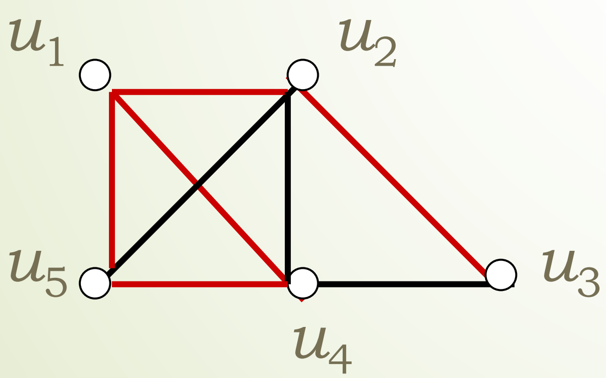
yes

What is the length?

3

Does it have any circuits?

no



$u_1, u_5, u_4, u_1, u_2, u_3$

Is it simple?

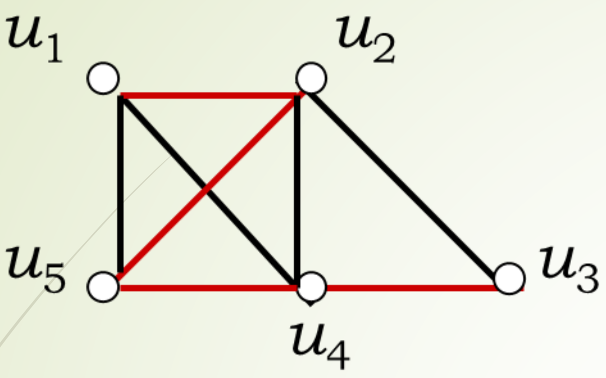
yes

What is the length?

5

Does it have any circuits?

Yes; u_1, u_5, u_4, u_1



u_1, u_2, u_5, u_4, u_3

Is it simple?

yes

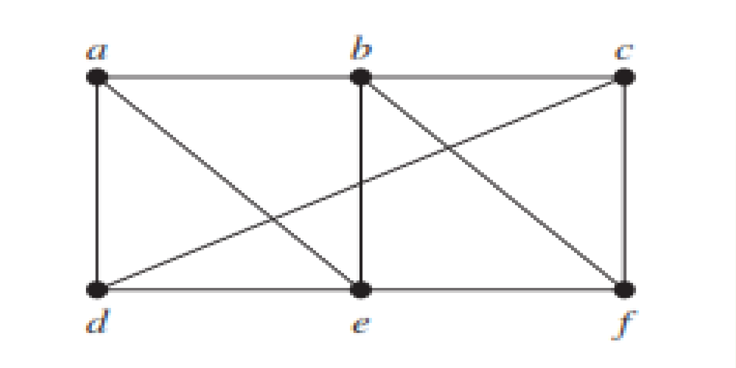
What is the length?

4

Does it have any circuits?

no

EXAMPLE 1 In the simple graph shown in Figure 1, a, d, c, f, e is a simple path of length 4, because $\{a, d\}$, $\{d, c\}$, $\{c, f\}$, and $\{f, e\}$ are all edges. However, d, e, c, a is not a path, because $\{e, c\}$ is not an edge. Note that b, c, f, e, b is a circuit of length 4 because $\{b, c\}$, $\{c, f\}$, $\{f, e\}$, and $\{e, b\}$ are edges, and this path begins and ends at b. The path a, b, e, d, a, b, which is of length 5, is not simple because it contains the edge $\{a, b\}$ twice. ◀



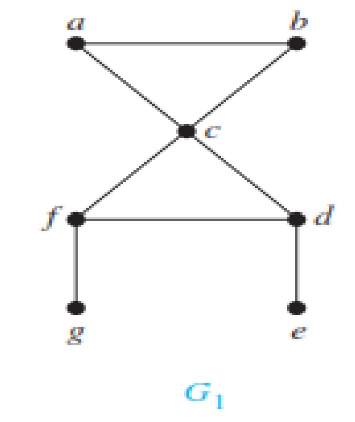
❑ Connectedness

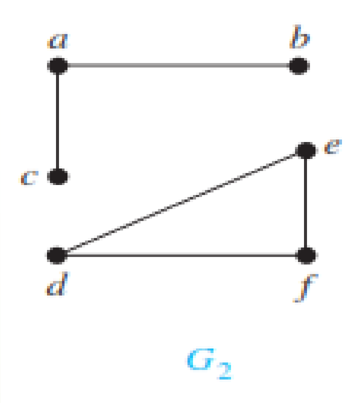
- An **undirected** graph is called connected if there is a path between every pair of distinct vertices of the graph.

- There is a simple path between every pair of distinct vertices of a connected undirected graph.

Examples

Are the following graphs connected?





- A graph that is not connected is the union of two or more disjoint connected subgraphs (called the connected components of the graph).

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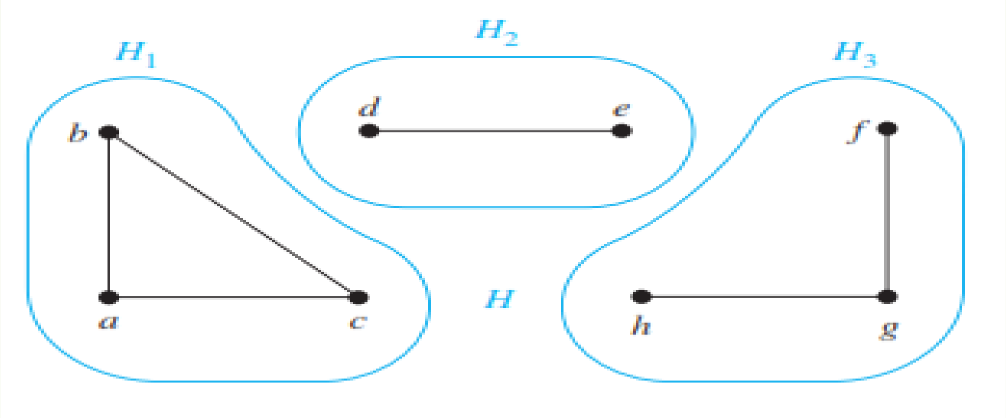
Example

What are the connected components of the following graph?



{a, b, c}, {d, e}, {f, g, h}

Cut edges and vertices

- If one can remove a vertex (and all incident edges) and produce a graph with more connected components, the vertex is called a *cut vertex*.

- If removal of an edge creates more connected components the edge is called a *cut edge* or *bridge*.

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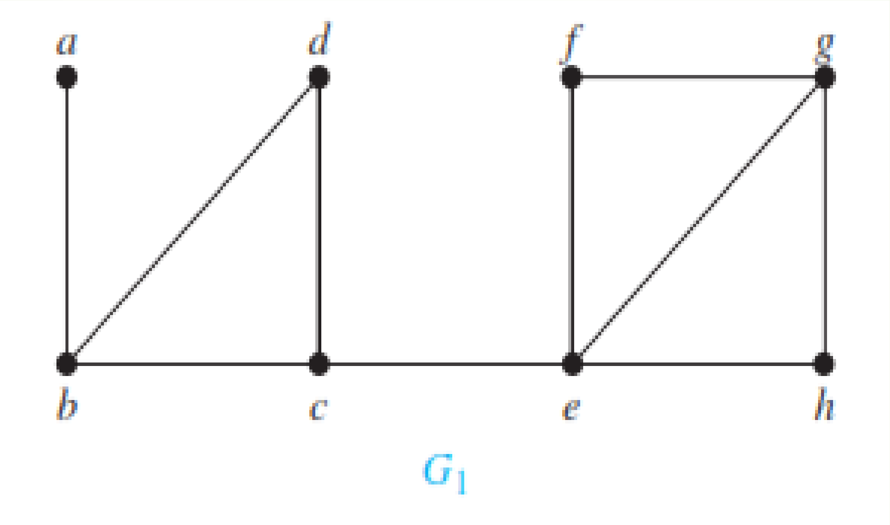
Example

Find the cut vertices and cut edges in the following graph.



*The cut vertices of G*1are b, c, and e. The removal of one of these vertices (and its adjacent edges) disconnects the graph. The cut edges are {a, b} and {c, e}. Removing either one of these edges disconnects G1

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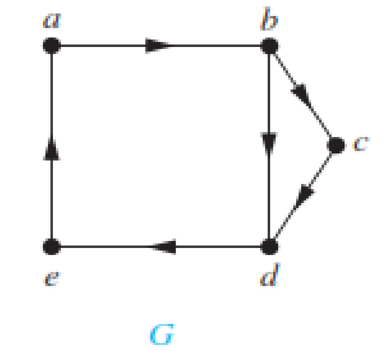
Connectedness in Directed Graphs

- A directed graph is ***strongly connected*** if there is a directed path between every pair of vertices a&b. (from a to b) (from b to a).

- A directed graph is ***weakly connected*** if there is a path between every pair of vertices in the underlying undirected graph, (i.e when the directions are disregarded).

Example

Is the following graph strongly connected? Is it weakly connected?

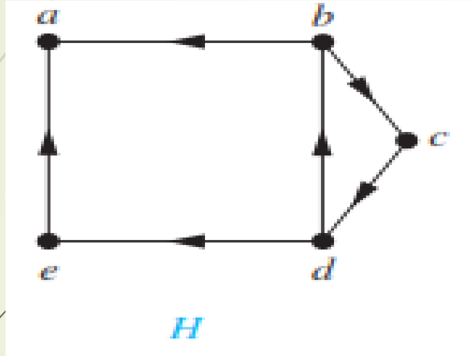


This graph is strongly connected. Why? Because there is a directed path between every pair of vertices.

If a directed graph is strongly connected, then it must also be weakly connected.

Example

Is the following graph strongly connected? Is it weakly connected?



This graph is not strongly connected. Why not? Because there is no directed path between *a* and *b*, *a* and *e*, etc. However, it *is* weakly connected. (Imagine this graph as an undirected graph.)

Counting Paths Between Vertices

THEOREM 2 Let G be a graph with adjacency matrix $\mathbf{A}$ with respect to the ordering $v_1, v_2, \ldots, v_n$ of the vertices of the graph (with directed or undirected edges, with multiple edges and loops allowed). The number of different paths of length r from v_i to v_j, where r is a positive integer, equals the (i, j)th entry of $\mathbf{A}^r$.

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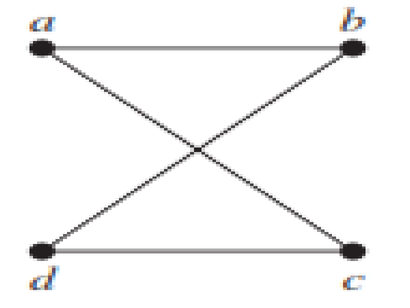
Example

How many paths of length four are there from a to d in the simple graph G ?



$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

Solution:

the number of paths of length four from a to d is the $(1, 4)$th entry of $\mathbf{A}^4$.

$$\mathbf{A}^4 = \begin{bmatrix} 8 & 0 & 0 & \underline{8} \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix},$$

there are exactly eight paths of length four from a to d. By inspection of the graph, we see that a, b, a, b, d; a, b, a, c, d; a, b, d, b, d; a, b, d, c, d; a, c, a, b, d; a, c, a, c, d; a, c, d, b, d; and a, c, d, c, d are the eight paths from a to d. ◀

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Lecture 6