CH.9 Relations

Lecture 6

Dr. Mohamed Abdel-Aal

9.1 Relations and Their Properties

If we want to describe a relationship between elements of two sets A and B, we can use **ordered pairs** with their first element taken from A and their second element taken from B. Since this is a relation between **two sets**, it is called a **binary relation**.

Definition: Let A and B be sets. A binary relation from A to B is a subset of $A \times B$.

In other words, for a binary relation R we have $R \subseteq A \times B$. We use the notation *a R b* to denote that $(a, b) \in R$ and $a \not R b$ to denote that $(a, b) \notin R$.

Example: Let P be a set of people, C be a set of cars, and D be the relation describing which person drives which car(s).

P = {Carl, Suzanne, Peter, Carla},

C = {Mercedes, BMW, tricycle}

D = {(Carl, Mercedes), (Suzanne, Mercedes), (Suzanne, BMW), (Peter, tricycle)}

This means that Carl drives a Mercedes, Suzanne drives a Mercedes and a BMW, Peter drives a tricycle, and Carla does not drive any of these vehicles.

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Functions as Relations

You might remember that a **function** f from a set A to a set B assigns a unique element of B to each element of A.
The **graph** of f is the set of ordered pairs (a, b) such that $b = f(a)$.
Since the graph of f is a subset of $A \times B$, it is a **relation** from A to B.
Moreover, for each element **a** of A, there is exactly one ordered pair in the graph that has **a** as its first element.

Relations on a Set

Definition: A relation on the set A is a relation from A to A.

In other words, a relation on the set A is a subset of $A \times A$.

Example: Let A = {1, 2, 3, 4}. Which ordered pairs are in the relation
R = {(a, b) | a divides b} ?

R ={(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)}.

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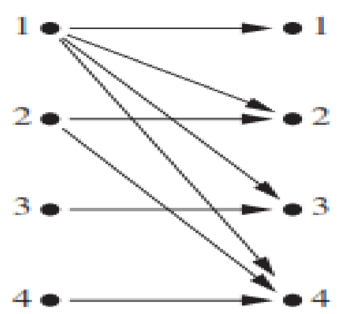

R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

Example: Consider these relations on the set of integers:

$$R_1 = \{(a, b) \mid a \le b\},$$
$$R_2 = \{(a, b) \mid a > b\},$$
$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\},$$
$$R_4 = \{(a, b) \mid a = b\},$$
$$R_5 = \{(a, b) \mid a = b + 1\},$$
$$R_6 = \{(a, b) \mid a + b \le 3\}.$$

Which of these relations contain each of the pairs (1, 1), (1, 2), (2, 1), (1, -1), and (2, 2)?

Solution: The pair (1, 1) is in R_1, R_3, R_4, and R_6; (1, 2) is in R_1 and R_6; (2, 1) is in R_2, R_5, and R_6; (1, −1) is in R_2, R_3, and R_6; and finally, (2, 2) is in R_1, R_3, and R_4. ◀

Example: Let A = {1, 2, 3, 4}. Which ordered pairs are in the relation

R = {(a, b) | a < b} ?

R = {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}.

Properties of Relations

We will now look at some useful ways to classify relations.

Definition: A relation R on a set A is called **reflexive** if $(a, a) \in R$ for every element $a \in A$.

Are the following relations on {1, 2, 3, 4} reflexive

R = {(1, 1), (1, 2), (2, 3), (3, 3), (4, 4)} No.

R = {(1, 1), (2, 2), (2, 3), (3, 3), (4, 4)} Yes.

R = {(1, 1), (2, 2), (3, 3)} No.

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Definitions:

A relation R on a set A is called **symmetric** If $(a, b) \in R$ implies $(b, a) \in R$ for all $a, b \in A$.

A relation R on a set A is called **antisymmetric** If $(a, b) \in R$ and $(b, a) \in R$ implies $a = b$.

Remark: Using quantifiers, we see that the relation R on the set A is *symmetric* *if*

$$\forall a \forall b((a, b) \in R \rightarrow (b, a) \in R).$$

Similarly, the relation R on the set A is *antisymmetric* if

$$\forall a \forall b(((a, b) \in R \land (b, a) \in R) \rightarrow (a = b)).$$

Which of the following relations are symmetric and which are antisymmetric?

Example 7 Consider the following relations on {1, 2, 3, 4}:

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$,

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,

$R_6 = \{(3, 4)\}$.

Solution: The relations R_2 and R_3 are symmetric, because in each case (b, a) belongs to the relation whenever (a, b) does. For R_3 , the only thing to check is that both (2, 1) and (1, 2) are in the relation. For R_3 , it is necessary to check that both (1, 2) and (2, 1) belong to the relation, and (1, 4) and (4, 1) belong to the relation. The reader should verify that none of the other relations is symmetric. This is done by finding a pair (a, b) such that it is in the relation but (b, a) is not.

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R_4, R_5, and R_6 are all antisymmetric. For each of these relations there is no pair of elements a and b with a = b such that both (a, b) and (b, a) belong to the relation. The reader should verify that none of the other relations is antisymmetric. This is done by finding a pair (a, b)with a ≠ b such that (a, b) and (b, a) are both in the relation.

Example

Is the “divides” relation on the set of positive integers symmetric?
Is it antisymmetric?

Solution: *This relation is not symmetric because 1| 2, but 2 ∤ 1. It is antisymmetric, for if a* and b are positive integers with a | b and b | a, then a = b

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Definition: A relation R on a set A is called <u>transitive</u> if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for $a, b, c \in A$.

Remark: Using quantifiers we see that the relation R on a set A is transitive if we have $\forall a \forall b \forall c(((a, b) \in R \land (b, c) \in R) \rightarrow (a, c) \in R)$.

Example

Which of the relations in Example 7 are transitive?

R_4, R_5, and R_6 are transitive. For each of these relations, we can show that it is transitive by verifying that if (a, b) and (b, c) belong to this relation, then (a, c) also does.

Example Are the following relations on {1, 2, 3, 4} transitive?

R = {(1, 1), (1, 2), (2, 2), (2, 1), (3, 3)} Yes.

R = {(1, 3), (3, 2), (2, 1)} No.

R = {(2, 4), (4, 3), (2, 3), (4, 1)} No.

Combining Relations

Relations are sets, and therefore, we can apply the usual set operations to them.

Example

Let A ={1, 2, 3} and B ={1, 2, 3, 4}. The relations R_1 ={(1, 1), (2, 2), (3, 3)} and R_2 ={(1, 1), (1, 2), (1, 3), (1, 4)} can be combined to obtain

$$R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\},$$
$$R_1 \cap R_2 = \{(1,1)\},$$
$$R_1 - R_2 = \{(2,2),(3,3)\},$$
$$R_2 - R_1 = \{(1,2),(1,3),(1,4)\}.$$

Quis

List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$, where $(a, b) \in R$ if and only if

a) $a = b$.
b) $a + b = 4$.
c) $a > b$.
d) $a \mid b$.

9.2 n-ary Relations and Their Applications

In order to study an interesting application of relations, namely **databases**, we first need to generalize the concept of binary relations to **n-ary relations**.

Definition: Let $A_1, A_2, \ldots, A_n$ be sets. An **n-ary relation** on these sets is a subset of $A_1 \times A_2 \times \ldots \times A_n$.
The sets $A_1, A_2, \ldots, A_n$ are called the **domains** of the relation, and n is called its **degree**

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Example

Let R = {(a, b, c) | $a = 2b \wedge b = 2c$ with a, b, $c \in \mathbf{N}$}

What is the degree of R?

The degree of R is 3, so its elements are triples.

What are its domains?

Its domains are all equal to the set of integers.

Is (2, 4, 8) in R?

No.

Is (4, 2, 1) in R?

Yes.

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Example

Let R be the relation on $\mathbf{N} \times \mathbf{N} \times \mathbf{N}$ consisting of triples (a, b, c), where a, b, and c are integers with $a < b < c$. Then $(1, 2, 3) \in R$, but $(2, 4, 3) \notin R$. The degree of this relation is 3. Its domains are all equal to the set of natural numbers.

Example

Let R be the relation on $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}^+$ consisting of triples (a,b,m), where a, b, and *m* are integers with $m \geq 1$ and $a \equiv b \pmod{m}$. Then (8, 2, 3), (-1, 9, 5), and (14, 0, 7) all belong to R, but (7, 2, 3), (-2, -8, 5), and (11, 0, 6) do not belong to R because $8 \equiv 2 \pmod 3$, $-1 \equiv 9 \pmod 5$, and $14 \equiv 0 \pmod 7$, but $7 \not\equiv 2 \pmod 3$, $-2 \not\equiv -8 \pmod 5$, and $11 \not\equiv 0 \pmod 6$.

This relation has degree 3 and its first two domains are the set of all integers and its third domain is the set of positive integers.

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9.3 Representing Relations

We already know different ways of representing relations. We will now take a closer look at two ways of representation: **Zero-one matrices** and **directed graphs**.

Representing Relations Using Matrices

If R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$, then R can be represented by the zero-one matrix $M_R = [m_{ij}]$ with

$m_{ij} = 1$, if $(a_i, b_j) \in R$, and

$m_{ij} = 0$, if $(a_i, b_j) \notin R$.

Note that for creating this matrix we first need to list the elements in A and B in a **particular, but arbitrary order**.

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Example: How can we represent the relation R = {(2, 1), (3, 1), (3, 2)} as a zero-one matrix?

Solution: The matrix M_R is given by

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

The 1s in M_R show that the pairs (2, 1), (3, 1), and (3, 2) belong to R. The 0s show that no other pairs belong to R.

- ❑ What do we know about the matrices representing a **relation on a set** (a relation from A to A) ?
- They are **square** matrices.
- ❑ What do we know about matrices representing **reflexive** relations?
- All the elements on the diagonal of such matrices Mref must be 1s.

$$M_{ref} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & 1 \end{bmatrix}$$

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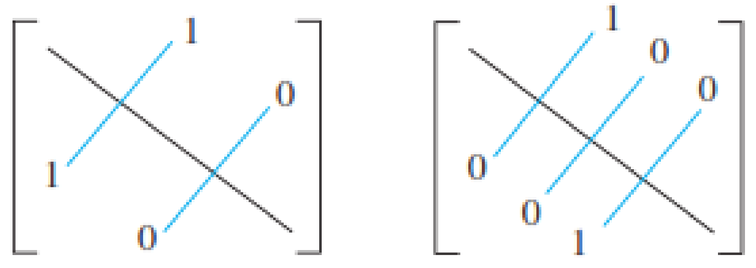
- What do we know about the matrices representing **symmetric relations**?
  - These matrices are symmetric, that is, $M_R = (M_R)^t$.



(a) Symmetric (b) Antisymmetric

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

symmetric matrix,
symmetric relation.

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

non-symmetric matrix,
non-symmetric relation.

Example

Suppose that the relation R on a set is represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Is R reflexive, symmetric, and/or antisymmetric?

- All the diagonal elements = 1, so R is reflexive.
- The lower left triangle of the matrix = the upper right triangle, so R is symmetric.
- To be antisymmetric, it must be the case that no more than one element in a symmetric position on either side of the diagonal = 1. But $M_{23} = M_{32} = 1$. So R is not antisymmetric.

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Let R be a binary relation on a set A and let M be the zero-one matrix for R.

R is *reflexive* iff $M_{ii} = 1$ for all i

R is *symmetric* iff M is a symmetric matrix, i.e., $M = M^T$

R is *antisymmetric* if $M_{ij} = 0$ or $M_{ji} = 0$ for all $i \neq j$

The Boolean operations **join** and **meet** (you remember?) can be used to determine the matrices representing the **union** and the **intersection** of two relations, respectively.

To obtain the **join** of two zero-one matrices, we apply the Boolean "or" function to all corresponding elements in the matrices.

To obtain the **meet** of two zero-one matrices, we apply the Boolean "and" function to all corresponding elements in the matrices.

Example: Let the relations R and S be represented by the matrices

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What are the matrices representing $R \cup S$ and $R \cap S$?

Solution: These matrices are given by

$$M_{R \cup S} = M_R \vee M_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad M_{R \cap S} = M_R \wedge M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Do you remember the **Boolean product** of two zero-one matrices?

Let $A = [a_{ij}]$ be an $m \times k$ zero-one matrix

and

$B = [b_{ij}]$ be a $k \times n$ zero-one matrix.

Then the **Boolean product** of A and B, denoted by AoB, is the m×n matrix with (i, j)th entry $[c_{ij}]$, where

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2i}) \vee \ldots \vee (a_{ik} \wedge b_{kj}).$$

$c_{ij} = 1$

if and only if at least one of the terms $(a_{in} \wedge b_{nj}) = 1$ for some n;

otherwise

$c_{ij} = 0.$

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In terms of the **relations**, this means that C contains a pair (x_i, z_j) if and only if there is an element y_n such that (x_i, y_n) is in relation A and (y_n, z_j) is in relation B.

Therefore, $C = B \circ A$ (**composite** of A and B).

This gives us the following rule:

$$M_{B \circ A} = M_A \odot M_B$$

In other words, the matrix representing the composite of relations A and B is the Boolean product of the matrices representing A and B.

Analogously, we can find matrices representing the powers of relations:

$M_{R^n} = M_R^{[n]}$ (n-th Boolean power).

Example: Find the matrix representing R^2, where the matrix representing R is given by

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

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Solution: The matrix for R^2 is given by

$$M_{R^2} = M_R^{[2]} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Example

Find the matrix representing the relations $S \circ R$, where the matrices representing R and S are

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Solution

The matrix for S∘R is

$$\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Quiz

Let R be the relation represented by the matrix

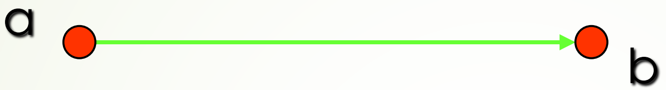
$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Find the matrix representing

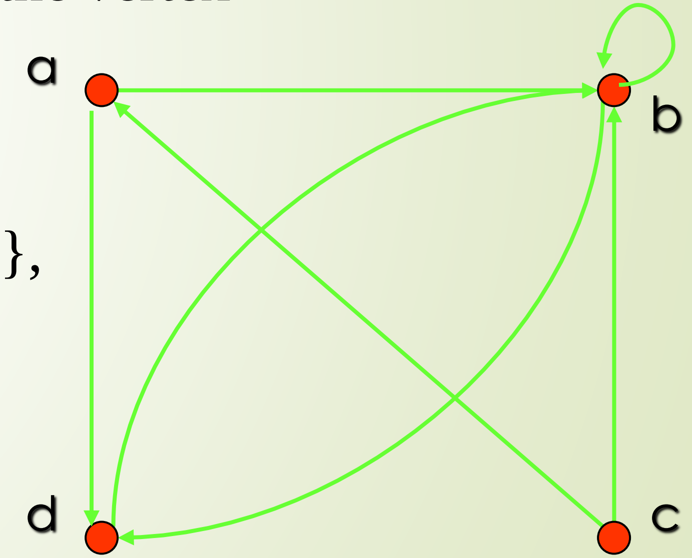
a) R^{-1}. **b)** $\overline{R}$. **c)** R^2.

Representing Relations Using Digraphs

Definition: A **directed graph**, or **digraph**, consists of a set **V** of **vertices** (or **nodes**) together with a set **E** of ordered pairs of elements of V called **edges** (or **arcs**).



The vertex a is called the initial vertex of the edge (a, b), and the vertex b is called the terminal vertex of this edge.



Example: Display the digraph (V,E) with V = {a, b, c, d}, E = {(a, b), (a, d), (b, b), (b, d), (c, a), (c, b), (d, b)}.

An edge of the form (b, b) is called a **loop**.

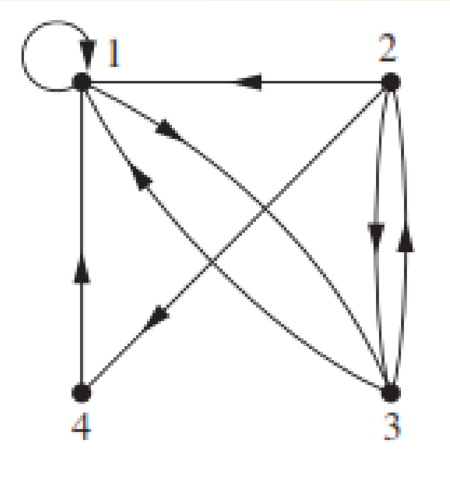
Obviously, we can represent any relation R on a set A by the digraph with A as its vertices and all pairs (a, b)∈R as its edges.

Vice versa, any digraph with vertices V and edges E can be represented by a relation on V containing all the pairs in E.

This **one-to-one correspondence** between relations and digraphs means that any statement about relations also applies to digraphs, and vice versa.

Example

What are the ordered pairs in the relation *R* represented by the directed graph to the left?



This digraph represents the relation

R = {(1,1), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (4,1)}

on the set {1, 2, 3, 4}.

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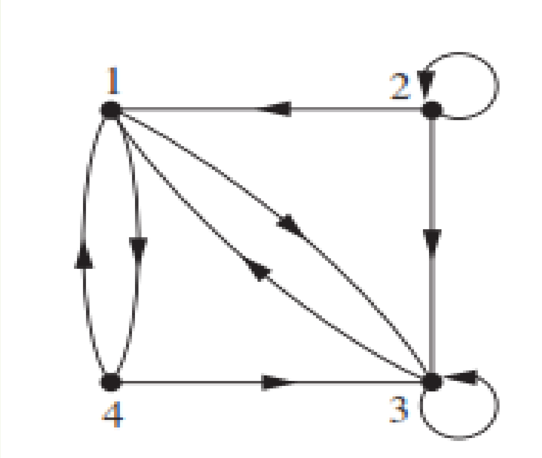
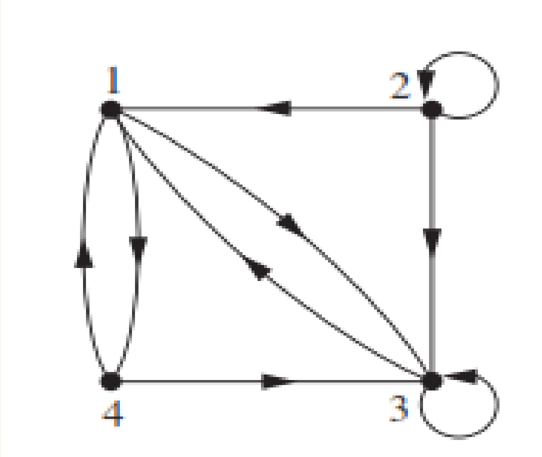
$R = \{(1,3), (1,4), (2,1), (2,2), (2,3), (3,1), (3,3), (4,1), (4,3)\}$



According to the digraph representing *R*:

- is (4,3) an ordered pair in *R*?
- is (3,4) an ordered pair in *R*?
- is (3,3) an ordered pair in *R*?



(4,3) is an ordered pair in *R*

(3,4) is not an ordered pair in *R* – no arrowhead pointing from 3 to 4

(3,3) is an ordered pair in *R* – loop back to itself

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A relation digraph can be used to determine whether the relation has various properties

Reflexive - must be a loop at every vertex.

Symmetric - for every edge between two distinct points there must be an edge in the opposite direction.

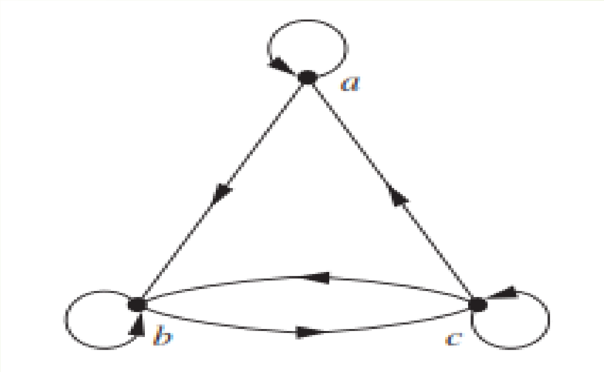
Antisymmetric - There are never two edges in opposite direction between two distinct points.

Transitive - If there is an edge from *x* to *y* and an edge from *y* to *z*, there must be an edge from *x* to *z*.

Example

According to the digraph representing *R*:

- is *R* reflexive?
- is *R* symmetric?
- is *R* antisymmetric?
- is *R* transitive?



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Solution:

•R is reflexive – there is a loop at every vertex

• R is not symmetric – there is an edge from *a* to *b* but not from *b* to *a*

• R is not antisymmetric – there are edges in both directions connecting *b* and *c*

• R is not transitive – there is an edge from *a* to *b* and an edge from *b* to *c*, but not from *a* to *c*

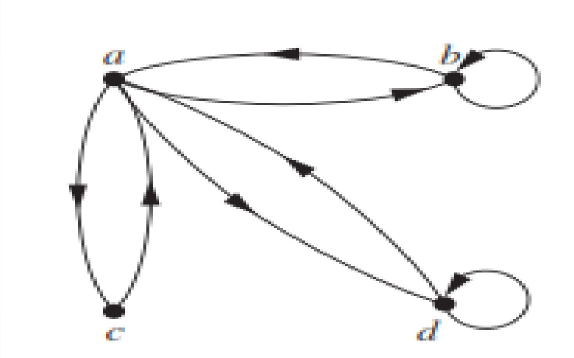
Example

According to the digraph representing *S*:

• is *S* reflexive?

• is *S* symmetric?

• is *S* antisymmetric?



•S is not reflexive – there aren't loops at every vertex

• S is symmetric – for every edge from one distinct vertex to another, there is a matching edge in the opposite direction

• S is not antisymmetric – there are edges in both directions connecting *a* and *b*

• S is not transitive – there is an edge from *c* to *a* and an edge from *a* to *b*, but not from *c* to *b*

9.4 Closures of Relations

Definition of Closure:

The *closure* of a relation R with respect to property P is the relation obtained by adding the minimum number of ordered pairs to R to obtain property P.

Properties: reflexive, symmetric, and transitive

Example

$A=\{1, 2, 3\}$

$R=\{(1,1), (1,2), (2,1), (3,2)\}$

Is R reflexive? Why?

What pairs do we need to make it reflexive?

(2,2), (3,3)

Reflexive closure of R= $\{(1,1), (1,2), (2,1), (3,2)\} \cup \{(2,2), (3,3)\}$ is reflexive.

Reflexive Closure

In terms of the digraph representation

Add loops to all vertices

In terms of the 0-1 matrix representation

Put 1's on the diagonal

Example: Symmetric closure

$A=\{1, 2, 3\}$
$R= \{(1,1), (1,2), (2,2), (2,3), (3,1), (3,2)\}$
Is R symmetric?
What pairs do we need to make it symmetric?

(2,1) and (1,3)
Symmetric closure of $R = \{(1,1), (1,2), (2,2), (2,3), (3,1), (3,2)\} \cup \{(2,1), (1,3)\}$

Symmetric Closure

Can be constructed by taking the union of a relation with its inverse.

In terms of the digraph representation

Add arcs in the opposite direction

In terms of the 0-1 matrix representation

Add 1's to the pairs across the diagonals that differ in value.

$$\begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{matrix} \quad \Rightarrow \quad \begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{matrix}$$

transitive Closure

THEOREM 3 Let $\mathbf{M}_R$ be the zero–one matrix of the relation R on a set with n elements. Then the zero–one matrix of the transitive closure R^* is

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} \vee \cdots \vee \mathbf{M}_R^{[n]}.$$

Example

Find the zero–one matrix of the transitive closure of the relation R where

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution

By Theorem 3, it follows that the zero–one matrix of R

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]}.$$

Because

$$\mathbf{M}_R^{[2]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_R^{[3]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

it follows that

$$\mathbf{M}_{R^*} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

9.5 Equivalence Relations

Equivalence relations are used to relate objects that are similar in some way.

Definition: A relation on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.

Two elements that are related by an equivalence relation R are called **equivalent**.

Example

Let R be the relation on the set of real numbers such that aRb if and only if a - b is an integer. Is R an equivalence relation?

Solution: Because a -a = 0 is an integer for all real numbers a, aRa for all real numbers a. Hence, R is reflexive. Now suppose that aRb. Then a -b is an integer, so b -a is also an integer. Hence, bRa. It follows that R is symmetric. If aRb and bRc, then a -b and b –c are integers. Therefore, a -c = (a -b) + (b -c) is also an integer. Hence, aRc. Thus, R is transitive. Consequently, R is an equivalence relation.

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Example: Suppose that R is the relation on the set of strings that consist of English letters such that aRb if and only if l(a) = l(b), where l(x) is the length of the string x. Is R an equivalence relation?

Solution:

- R is reflexive, because l(a) = l(a) and therefore aRa for any string a.
- R is symmetric, because if l(a) = l(b) then l(b) = l(a), so if aRb then bRa.
- R is transitive, because if l(a) = l(b) and l(b) = l(c), then l(a) = l(c), so aRb and bRc implies aRc. R is an equivalence relation.

Example

Show that the "divides" relation is the set of positive integers in not an equivalence relation.

we know that the "divides" relation is reflexive and transitive. However, we know that this relation is not symmetric (for instance, $2 \mid 4$ but $4 \nmid 2$). We conclude that the "divides" relation on the set of positive integers is not an equivalence relation.

Definition: Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the **equivalence class** of a. The equivalence class of a with respect to R is denoted by **$[a]_R$**. When only one relation is under consideration, we will delete the subscript R and write **[a]** for this equivalence class. If $b \in [a]_R$, b is called a **representative** of this equivalence class.

$$[a]_R = \{s \mid (a, s) \in R\}.$$

Example

What are the equivalence classes of 0 and 1 for congruence modulo 4?

The equivalence class of 0 contains all integers a such that $a \equiv 0 \pmod 4$. The integers in this class are those divisible by 4. Hence, the equivalence class of 0 for this relation is $[0]=\{\ldots,-8, -4, 0, 4, 8,\ldots\}$.

The equivalence class of 1 contains all the integers a such that $a \equiv 1 \pmod 4$. The integers in this class are those that have a remainder of 1 when divided by 4. Hence, the equivalence class of 1 for this relation is $[1]=\{\ldots,-7, -3, 1, 5, 9,\ldots\}$.

$[0]=\{\ldots,-8, -4, 0, 4, 8,\ldots\}$.
$[1]=\{\ldots,-7, -3, 1, 5, 9,\ldots\}$.

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Equivalence Classes and Partitions

THEOREM 1 Let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

(*i*) aRb (*ii*) $[a] = [b]$ (*iii*) $[a] \cap [b] \neq \emptyset$

Definition: A **partition** of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i, $i \in I$, forms a partition of S if and only if

(i) $A_i \neq \emptyset$ for $i \in I$

(ii) $A_i \cap A_j = \emptyset$, if $i \neq j$

(iii) $\cup_{i \in I} A_i = S$

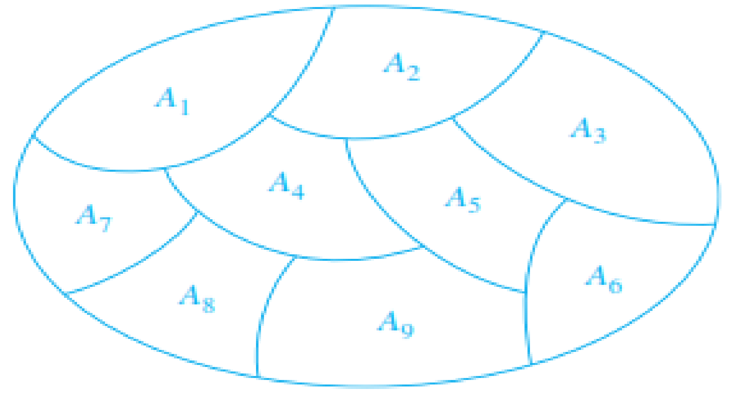

FIGURE 1 **A Partition of a Set.**

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Examples: Let S be the set {u, m, b, r, o, c, k, s}. Do the following collections of sets partition S ?

{{m, o, c, k}, {r, u, b, s}} yes

{{c, o, m, b}, {u, s}, {r}} no (k is missing).

{{b, r, o, c, k}, {m, u, s, t}} no (t is not in S).

{{u, m, b, r, o, c, k, s}} yes.

{{b, o, o, k}, {r, u, m}, {c, s}} yes ({b,o,o,k} = {b,o,k}).

{{u, m, b}, {r, o, c, k, s}, ∅} no (∅ not allowed).

Theorem: Let R be an equivalence relation on a set S. Then the **equivalence classes** of R form a **partition** of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

Example

Suppose that S ={1, 2, 3, 4, 5, 6}. The collection of sets A_1={1, 2, 3}, A_2 ={4, 5}, and A_3 ={6} forms a partition of S, because these sets are disjoint and their union is S.

Another example: Let R be the relation $\{(a, b) \mid a \equiv b \pmod 3\}$ on the set of integers. Is R an equivalence relation?

Yes, R is reflexive, symmetric, and transitive.

What are the equivalence classes of R ?

{{…, -6, -3, 0, 3, 6, …}, {…, -5, -2, 1, 4, 7, …}, {…, -4, -1, 2, 5, 8, …}}

Example

What are the sets in the partition of the integers arising from congruence modulo 4?

$$[0]_4 = \{\ldots, -8, -4, 0, 4, 8, \ldots\},$$
$$[1]_4 = \{\ldots, -7, -3, 1, 5, 9, \ldots\},$$
$$[2]_4 = \{\ldots, -6, -2, 2, 6, 10, \ldots\},$$
$$[3]_4 = \{\ldots, -5, -1, 3, 7, 11, \ldots\}.$$

These congruence classes are disjoint, and every integer is in exactly one of them. In other words, as Theorem 2 says, these congruence classes form a partition.

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