Lecture 5

CH.2 Basic Structures: Sets, Functions, Sequences and Summations

- Functions
  - Introduction
  - One-to-One and Onto Functions
  - Inverse Functions and Compositions of Functions
- Sequences and Summations

Function

Let A and B be nonempty sets. A ***function*** *f from A to B is an assignment of* ***exactly*** *one* element of B to **each** element of A. We write *f(a) = b* if b is the unique element of B assigned by the function *f* to the element *a* of A. If *f* is a function from A to B, we write $f : A \to B$.

Functions are sometimes also called mappings or transformations

If *f* is a function from A to B, we say that A is the **domain** of *f* and B is the **codomain** of *f*. If *f(a) = b*, we say that *b* is the **image** of *a* and *a* is a preimage of *b*. The range, or image, of *f* is the set of all images of elements of A. Also, if *f* is a function from A to B, we say that *f* maps A to B.

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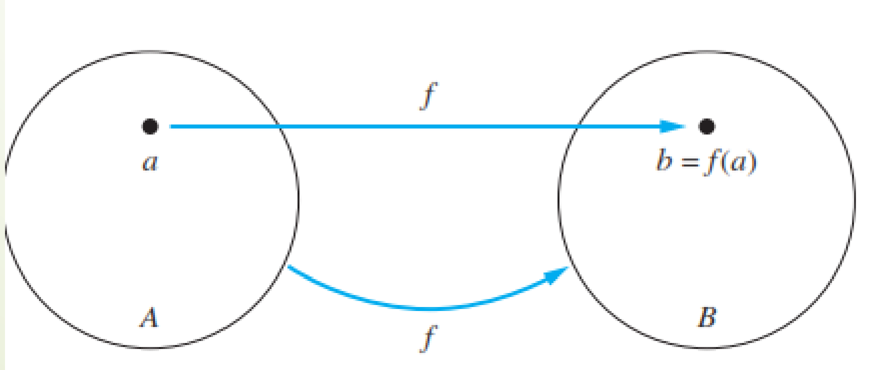

Examples

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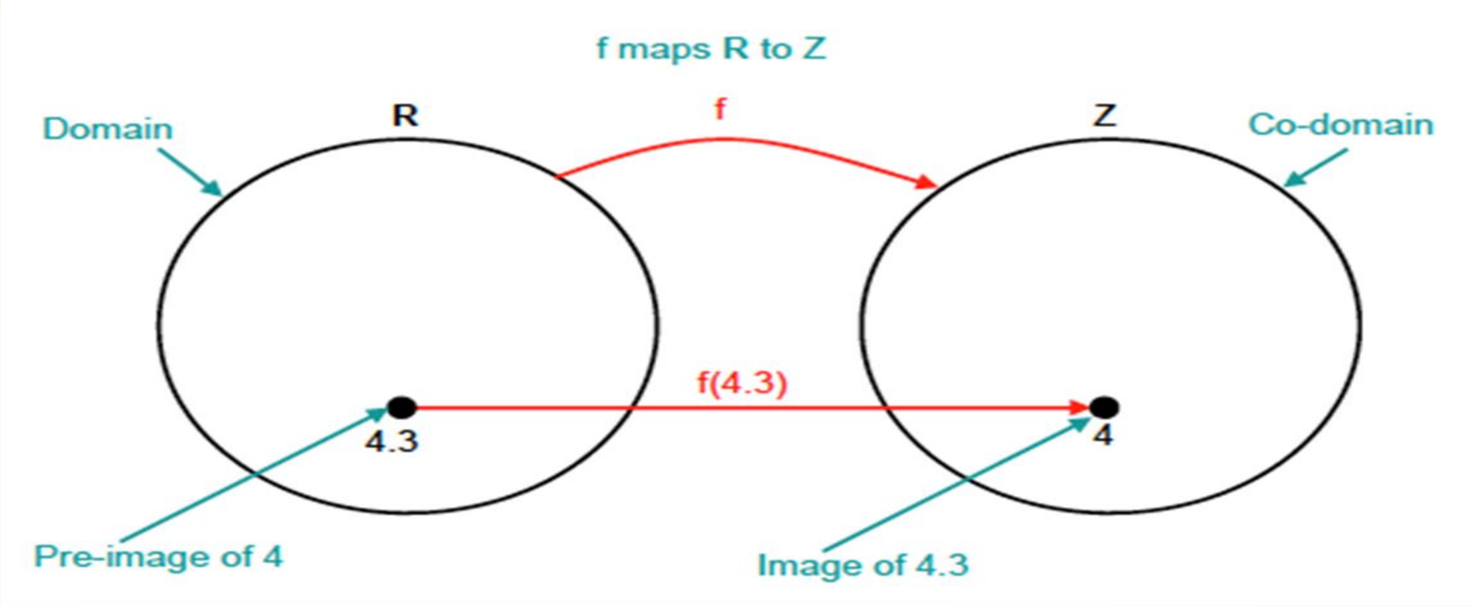


Example of the floor function

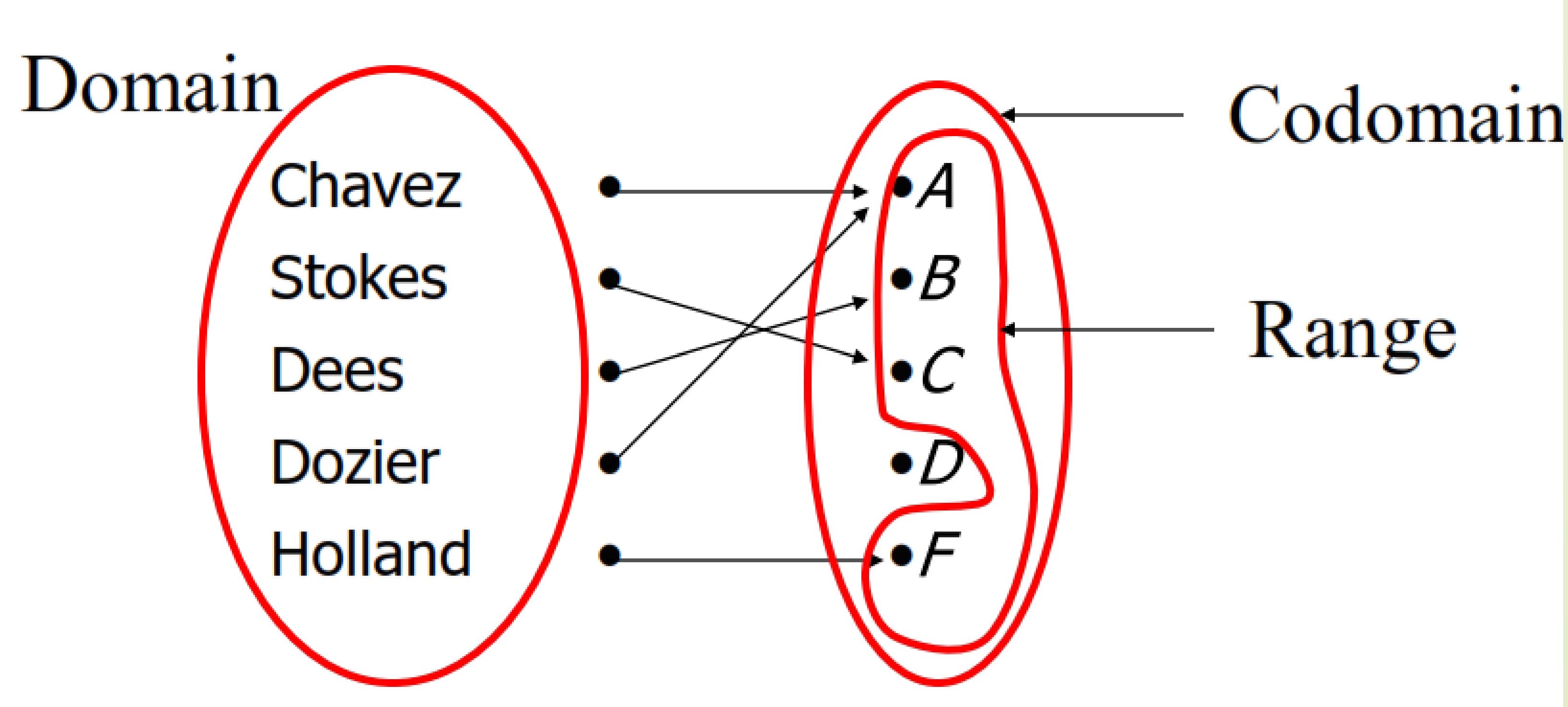
- f: $Z \to Z$, assigns the square of an integer to its integer, $f(x)=x^2$

Domain : the set of all integers

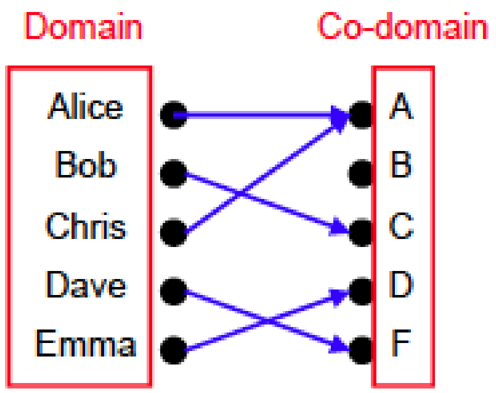
Codomain : set of all integers

Range: all integers that are perfect squares, i.e., {0, 1, 4, 9, …}

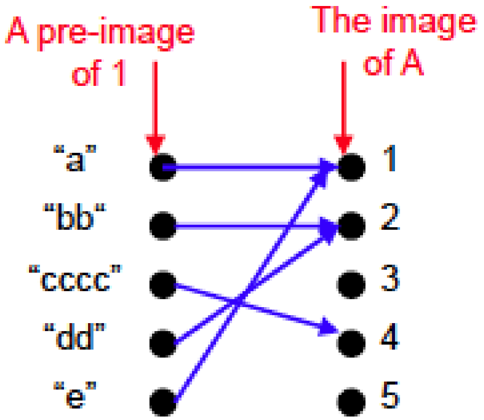
Suppose that each student in a class is assigned a letter grade from the set {***A, B, C, D, F***}. ***Let g*** be the function that assigns a grade to a student.



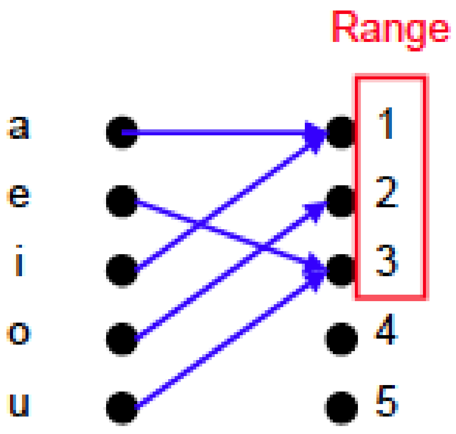
EXAMPLES



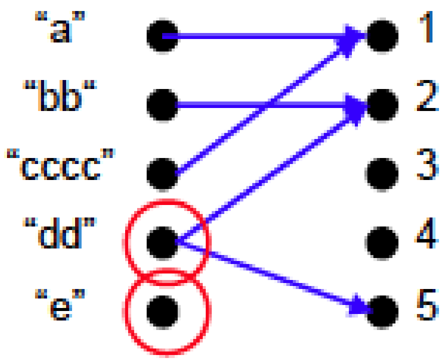
A class grade function



A string length function



Some function…



Not a valid function!
Also not a valid function!

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- Specify a function by
  - Domain
  - Codomain
  - Mapping of elements
- Two functions are equal if they have
  Same domain, codomain, mapping of elements

Quiz (1)

1. Why is f not a function from R to R if

a) $f(x) = 1/x$?
b) $f(x) = \sqrt{x}$?

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- Two real-valued functions with the same domain can be added and multiplied

Let f_1 and f_2 be functions from A to $\boldsymbol{R}$, then f_1+f_2, and $f_1\ f_2$ are also functions from A to $\boldsymbol{R}$ defined by

$$(f_1+f_2)(x)= f_1(x)+f_2(x)$$

$$(f_1f_2)(x)= f_1(x)\ f_2(x)$$

Note that the functions f_1+f_2 and f_1f_2 at x are defined in terms f_1 and f_2 at x

EXAMPLE

Let f_1 and f_2 be functions from $\mathbf{R}$ to $\mathbf{R}$ such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. What are the functions $f_1 + f_2$ and $f_1 f_2$?

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From the definition of the sum and product of functions, it follows that

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x$$

and

$$(f_1 f_2)(x) = x^2(x - x^2) = x^3 - x^4.$$

Let *f* be a function from A to B and let *S* be a subset of A. The *image of S under the function f* :
is the subset of *B* that consists of the images of the elements of S.
We denote the image of S by *f(S)*, so

$$f(S) = \{t \mid \exists s \in S\ (t = f(s))\}.$$

We also use the shorthand $\{f(s) \mid s \in S\}$ to denote this set.

Remark: *f(S)* denotes a set, and not the value of the function *f* for the set S.

EXAMPLES

Lecture 4

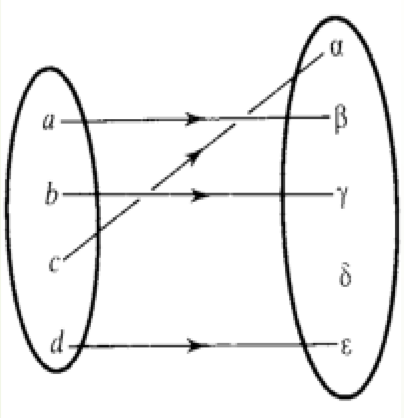
Let A ={a, b, c, d, e} and B ={1, 2, 3, 4} with f(a) = 2, f(b) = 1, f(c) = 4, f(d) = 1, and f(e) = 1. The image of the subset S ={b, c, d} is the set *f(S)* ={1, 4}.

• One-to-One and Onto Functions

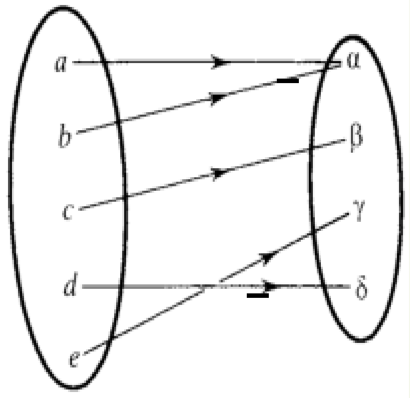
Injections and Surjections

In this section we consider two special kinds of functions: ‘injections’ and ‘surjections’. a function $f : A \longrightarrow B$ *can be such that:*

(i) different elements of the domain may have the same image in the codomain;
(ii) there may be elements of the codomain which are not the image of any element of the domain.



(a)



(b)

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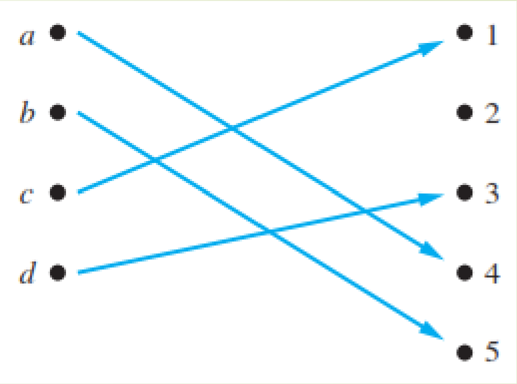
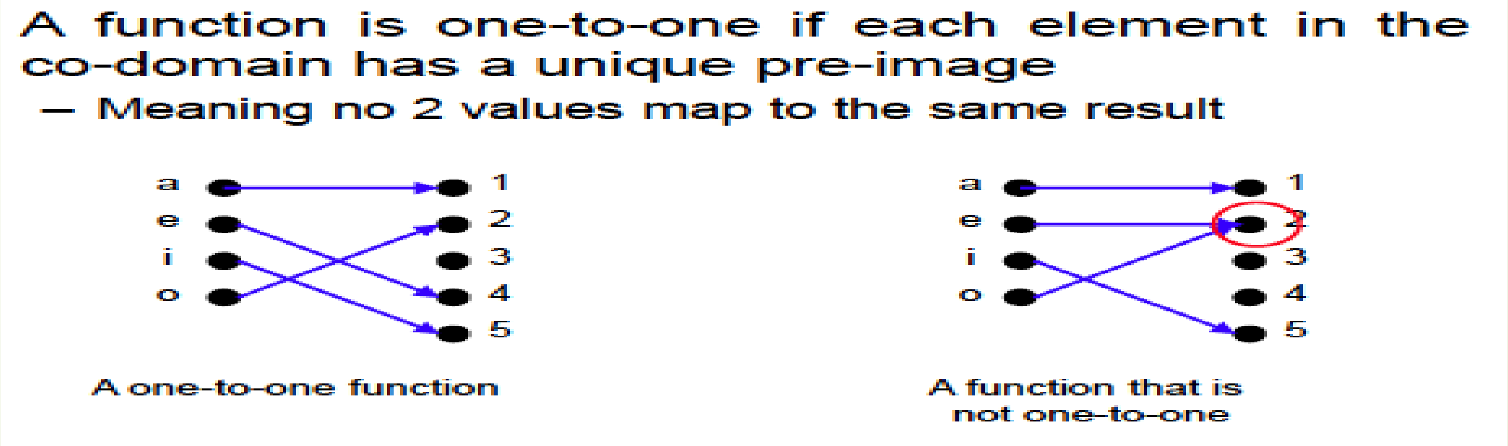
DEFINITION

A function f is said to be *one-to-one*, *or an* **injunction**, *if and only if* $f(a) = f(b)$ *implies that* $a = b$ for all a and b in the domain of f. A function is said to be *injective if it is one-to-one.*



Remark: We can express that f is one-to-one using quantifiers as $\forall a\ \forall b\ (f(a) = f(b) \rightarrow a = b)$
or using the contrapositive
equivalently $\forall a\ \forall b(a \neq b \rightarrow f(a) \neq f(b))$,

- Every element of B is the image of a unique element of A



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EXAMPLES

- Determine whether the function f(x) = x^2 from the set of integers to the set of integers is one-to-one.

The function f(x) = x^2 is **not** one-to-one because, for instance,

f(1) = f(-1) = 1, but 1 ≠ -1.

- Determine whether the function f(x) = x^2 from the set of positive integers to the set of integers is one-to-one ????.

- Determine whether the function f*(x)* = *x* + *1* from the set of real numbers to itself is one-to one.

The function f(x) = x + 1 is a one-to-one function. To demonstrate this, note that x + 1 ≠ y + 1 when x ≠ y

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Increasing/decreasing functions

Increasing (decreasing): if $f(x) \leq f(y)$ ($f(x) \geq f(y)$), whenever $x<y$ and x, y are in the domain of f

Strictly increasing (decreasing): if $f(x)<f(y)$ ($f(x) > f(y)$) whenever $x<y$, and x, y are in the domain of f

A function that is either strictly increasing or decreasing must be one-to-one

Remark: A function f is increasing if $\forall x \forall y(x < y \to f(x) \leq f(y))$, strictly increasing if $\forall x \forall y(x < y \to f(x) < f(y))$, decreasing if $\forall x \forall y(x < y \to f(x) \geq f(y))$, and strictly decreasing if $\forall x \forall y(x < y \to f(x) > f(y))$, where the universe of discourse is the domain of f.

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Onto functions

A function *f* from A to B is called *onto*, *or a surjection*, *if and only if for every element* b ∈ B there is an element *a* ∈ *A* with f(a) = b. A function *f* is called *surjective if it is onto*.

Remark: A function *f* is onto if

$$\forall y\ \exists x\ (f(x) = y),$$

where the domain for *x* is the domain of the function and the domain for y is the codomain of the function.

Codomain = range!

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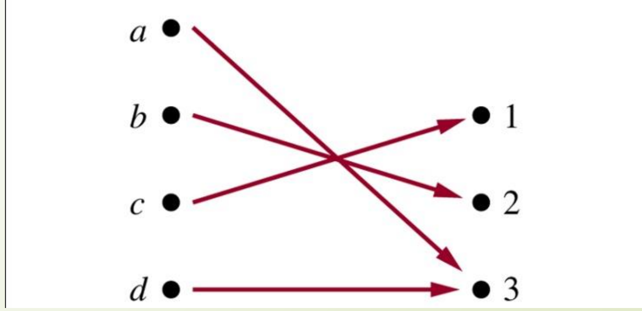
EXAMPLES

f maps from {a, b, c, d} to {1, 2, 3}, is f onto?



A function is onto if each element in the co-domain is an image of some pre-image

– Meaning all elements in the right are mapped to

An onto function

A function that is not onto

- Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

Is it true that $\forall y \ \exists x \ (x^2=y)$?

The function f is **not** onto because there is no integer x with $x^2 = -1$, for instance.

-1 is one of the possible values of y, but there does not exists an x such that $x^2 = -1$

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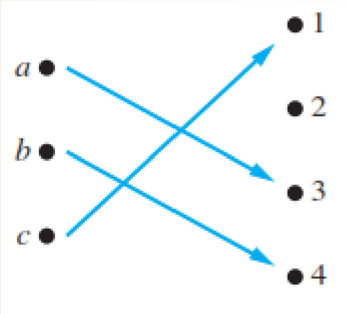
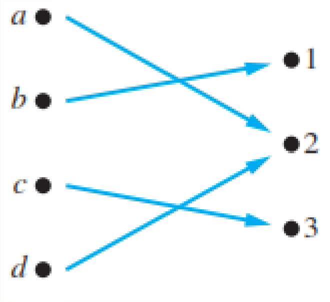
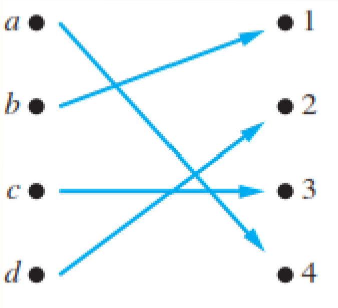
- Is *f(x)=x+1* from the set of integers to the set of integers onto?
  - It is onto, as for each integer y there is an integer *x* such that *f(x) =y*
  - To see this, f(x) = y *iff* x+1 = y, which holds if and only if x = y-1



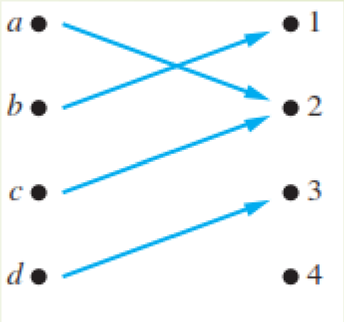
(a) One-to-one, not onto



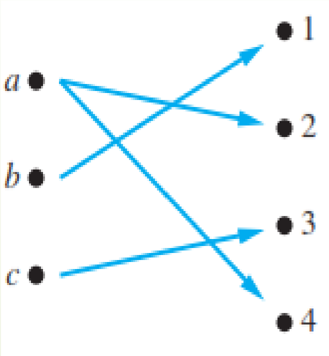
(b) Onto, not one-to-one



(c) One-to-one, and onto



(d) Neither one-to-one nor onto



(e) Not a function

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One-to-one correspondence

The function f is a *one-to-one correspondence*, *or a bijection*, *if it is both one-to-one and* onto. We also say that such a function is *bijective.*

- Let *f* be the function from {a, b, c, d} to {1, 2, 3, 4} with f(a)=4, f(b)=2, f(c)=1, and f(d)=3, is f bijective?
  - It is one-to-one as no two values in the domain are assigned the same function value
  - It is onto as all four elements of the codomain are images of elements in the domain

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Identity function:

It is one-to-one and onto

$$\iota_A : A \to A, \iota_A(x) = x, \forall x \in A$$

we summarize what needs be to shown to establish whether a function is one-to-one and whether it is onto.

Suppose that $f : A \to B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x \neq y$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

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Quiz (2)

Determine whether each of these functions is a bijection from **R** to **R**.

a) $f(x) = 2x + 1$

b) $f(x) = x^2 + 1$

c) $f(x) = x^3$

• Inverse Functions and Compositions of Functions

Consider a one-to-one correspondence f from A to B

Since f is onto, every element of B is the image of some element in A

Since f is one-to-one, every element of B is the image of a unique element of A

Thus, we can define a new function from B to A that reverses the correspondence given by f

DEFINITION

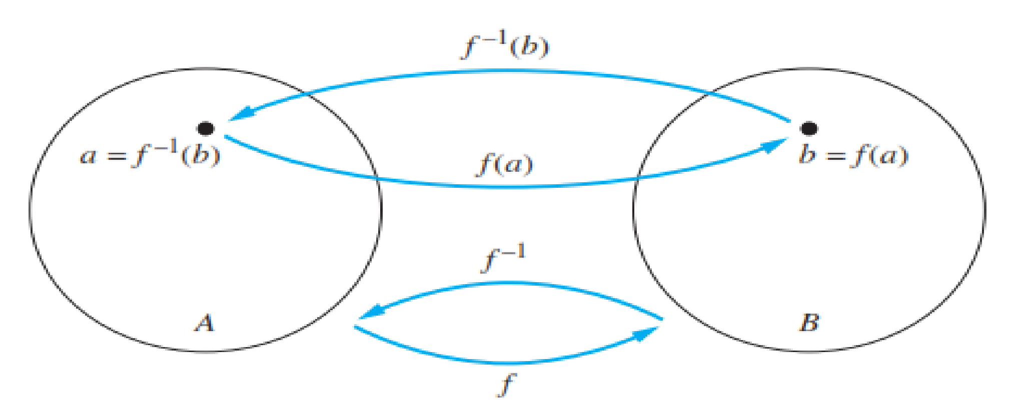
Let f be a one-to-one correspondence from the set A to the set B

The **inverse function** of f is the function that assigns an element b belonging to B the unique element a in A such that $f(a)=b$

Denoted by f^{-1}, hence $f^{-1}(b)=a$ when $f(a)=b$

Note f^{-1} is not the same as $1/f$

- A one-to-one correspondence is called **invertible**



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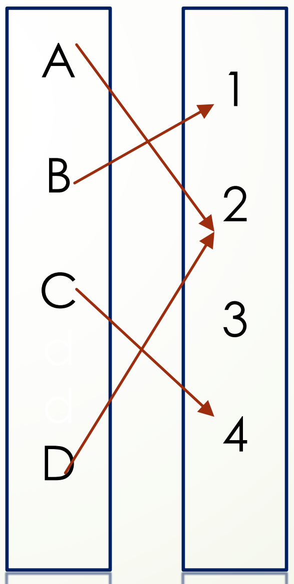
Why can't we invert such a function?

We cannot assign to each element *b in the* codomain a unique element a *in the domain* such that *f(a) = b, because:*

– For some *b there is either*

• More than one *a*

• No such *a*



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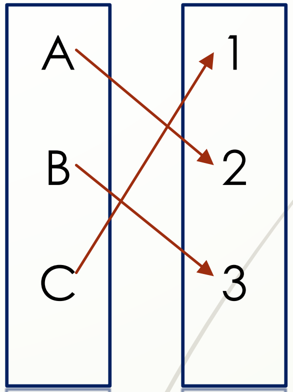
EXAMPLES



- *f* is a function from {a, b, c} to {1, 2, 3} with f(a)=2, f(b)=3, f(c)=1. Is it invertible? What is it its inverse?

The function *f* is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f ,so
$f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

- Let *f*: *Z*→*Z* such that *f*(*x*) = *x*+1, Is *f* invertible? If so, what is its inverse?

$$y=x+1,\ x=y-1,\ f^{-1}(y) = y-1$$

- Let *f*: *R*→*R* with $f(x)=x^2$, Is it invertible?

Since f(2)=f(-2)=4, f is not one-to-one, and so not invertible

- Sometimes we restrict the domain or the codomain of a function or both, to have an invertible function

  The function $f(x)=x^2$, from R^+ to R^+ is

  - one-to-one : If $f(x) = f(y)$, then $x^2 = y^2$, so

  $x^2 - y^2 = (x - y)(x + y)$ then x + y = 0 or x-y = 0, so x=-y or x=y Because both x and y are nonnegative, we must have x=y

  onto: $y= x^2$, every non-negative real number has a square root inverse function: $x = f^{-1}(y) = \sqrt{y}$.

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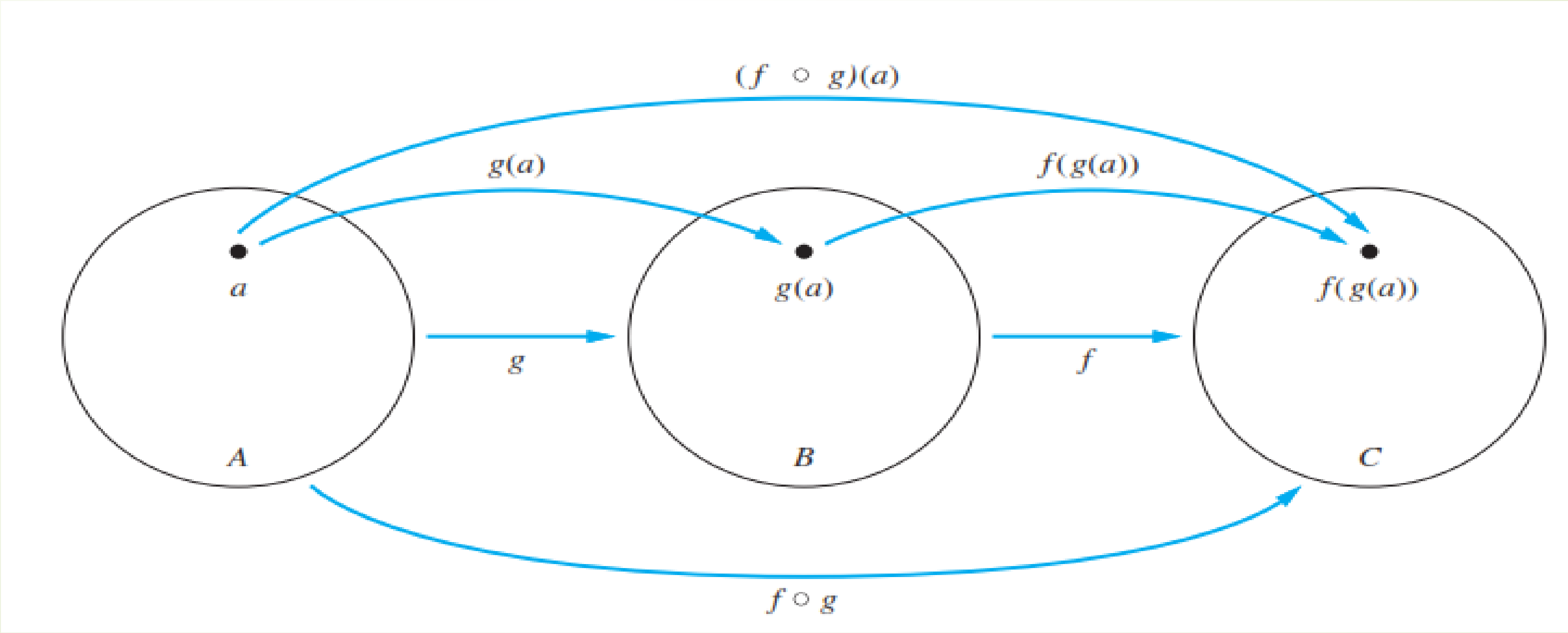
Composition of functions

Let g be a function from A to B and f be a function from B to C, the *composition of the functions f and g*, denoted by $f \circ g$, is defined by

$$(f \circ g)(a)=f(g(a))$$

- First apply g to a to obtain $g(a)$
- Then apply f to $g(a)$ to obtain $(f \circ g)(a)=f(g(a))$



Note $f \circ g$ cannot be defined unless the range of g is a subset of the domain of f

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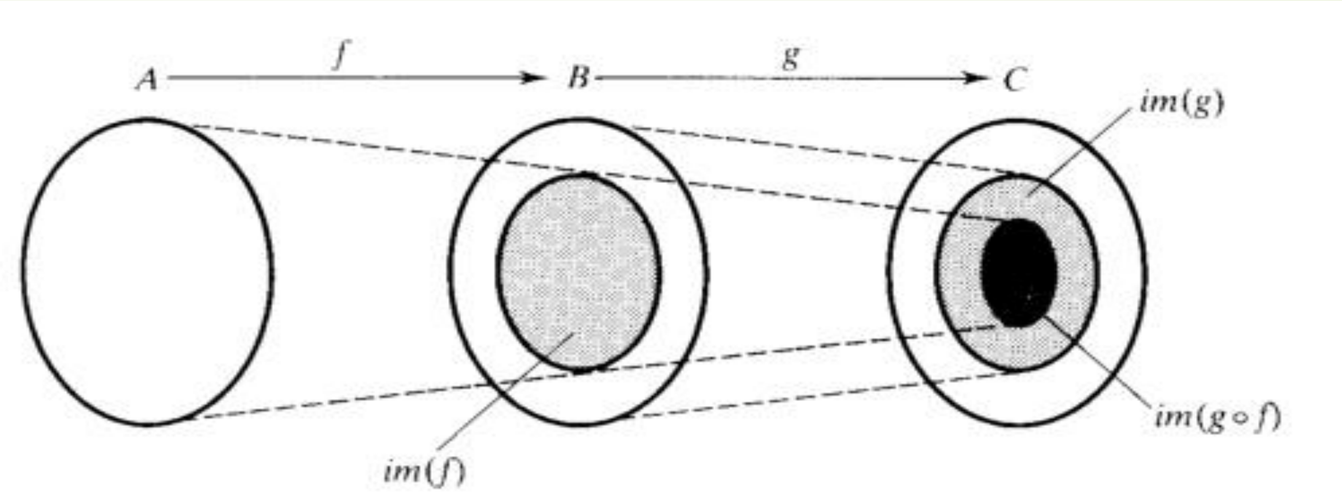

EXAMPLES

- g: {a, b, c} → {a, b, c}, g(a)=b, g(b)=c, g(c)=a, and f:{a,b,c} →{1,2,3}, f(a)=3, f(b)=2, f(c)=1.

What are $f \circ g$ and $g \circ f$?

(f∘g)(a)=f(g(a))=f(b)=2,

(f∘g)(b)=f(g(b))=f(c)=1,

(f∘g)(c)=f(a)=3

(g∘f)(a)=g(f(a))=g(3) not defined. g∘f is not defined

Note that *g* ∘ *f* is not defined, because the range of *f* is not a subset of the domain of *g*

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- f(x)=2x+3, g(x)=3x+2. What are $f \circ g$ and $g \circ f$?
  $(f \circ g)(x)=f(g(x))=f(3x+2)=2(3x+2)+3=6x+7$
  $(g \circ f)(x)=g(f(x))=g(2x+3)=3(2x+3)+2=6x+11$

Note that $f \circ g$ and $g \circ f$ are defined in this example, but they are not equal The commutative law does not hold for composition of functions

Composition of Inverses

$f \circ f^{-1}$ form an identity function in any order

Let $f: A \rightarrow B$ with $f(a)=b$

Suppose f is one-to-one correspondence from A to B
Then f^{-1} is one-to-one correspondence from B to A
The inverse function reverses the correspondence of f, so $f^{-1}(b)=a$ when $f(a)=b$, and $f(a)=b$ when $f^{-1}(b)=a$

$$(f^{-1} \circ f)(a)=f^{-1}(f(a))=f^{-1}(b)=a,$$

and

$$(f \circ f^{-1})(b)=f(f^{-1})(b))=f(a)=b$$

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$f^{-1} \circ f = \iota_A,\ f \circ f^{-1} = \iota_B,\ \iota_A, \iota_B$ are identity functions of A and B

$(f^{-1})^{-1} = f$

Important functions – Floor

Let x be a real number. The floor function is the closest integer less than or equal to x.

Examples

$\lfloor ½ \rfloor = 0$

$\lfloor -½ \rfloor = ?$

$\lfloor 3.1 \rfloor = ?$

$\lfloor 7 \rfloor = ?$

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Important functions – Ceiling

Let x be a real number. The ceiling function is the closest integer greater than or equal to x.

Examples

$\lceil 1/2 \rceil = 1$

$\lceil -1/2 \rceil = ?$

$\lceil 3.1 \rceil = ?$

$\lceil 7 \rceil = ?$

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \le x < n+1$
(1b)	$\lceil x \rceil = n$ if and only if $n-1 < x \le n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x-1 < n \le x$
(1d)	$\lceil x \rceil = n$ if and only if $x \le n < x+1$
(2)	$x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x + n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x + n \rceil = \lceil x \rceil + n$

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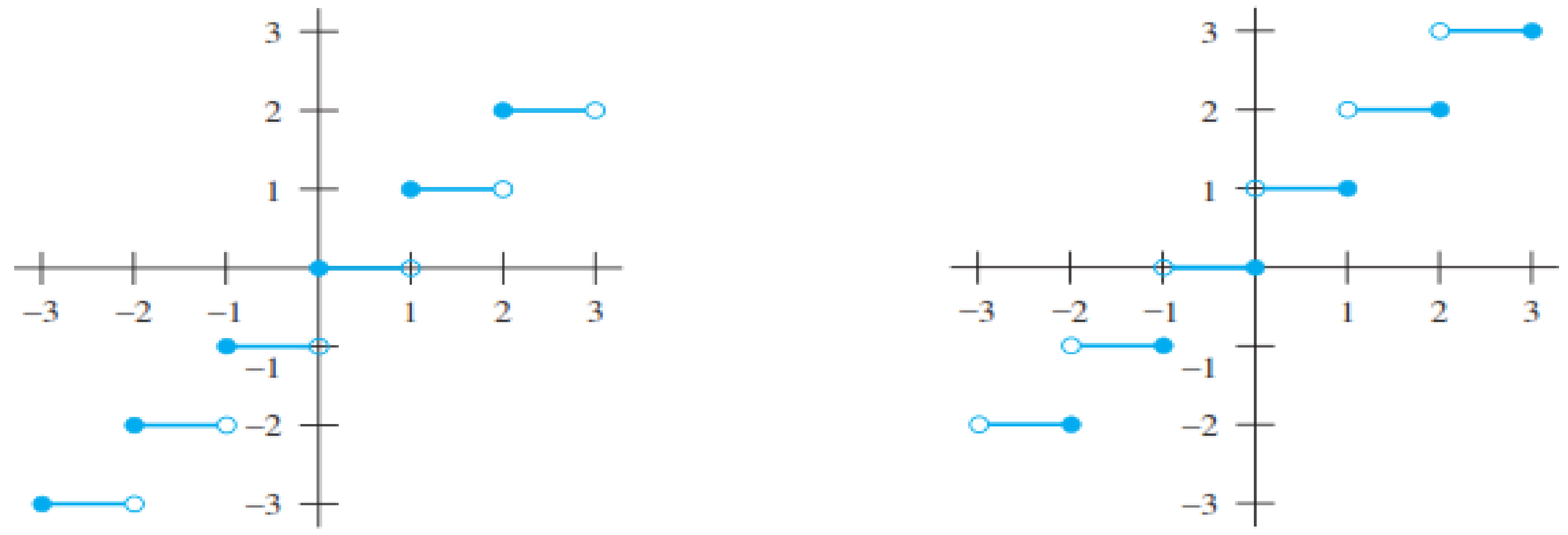

(a) $y = \lfloor x \rfloor$ (b) $y = \lceil x \rceil$

Graphs of the (a) Floor and (b) Ceiling Functions.

Quiz (3)

8. Find these values.

a) $\lfloor 1.1 \rfloor$

b) $\lceil 1.1 \rceil$

c) $\lfloor -0.1 \rfloor$

d) $\lceil -0.1 \rceil$

e) $\lceil 2.99 \rceil$

f) $\lceil -2.99 \rceil$

g) $\lfloor \frac{1}{2} + \lceil \frac{1}{2} \rceil \rfloor$

h) $\lceil \lfloor \frac{1}{2} \rfloor + \lceil \frac{1}{2} \rceil + \frac{1}{2} \rceil$

Prove that if n is an integer, then $\lfloor n/2 \rfloor = n/2$ if n is even and $(n - 1)/2$ if n is odd.

The Graphs of Functions

Let f be a function from the set A to the set B.
The *graph of the function f is the set of* ordered pairs

$$\{(a, b) \mid a \in A \text{ and } f(a) = b\}.$$

EXAMPLES

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1. Display the graph of the function $f(n) = 2n + 1$ from the set of integers to the set of integers.

The graph of f is the set of ordered pairs of the form $(n, 2n + 1)$, where n is an integer.

2. Display the graph of the function $f(x) = x^2$ from the set of integers to the set of integers.

The graph of f is the set of ordered pairs of the form $(x, f(x)) = (x, x^2)$, where x is an integer.

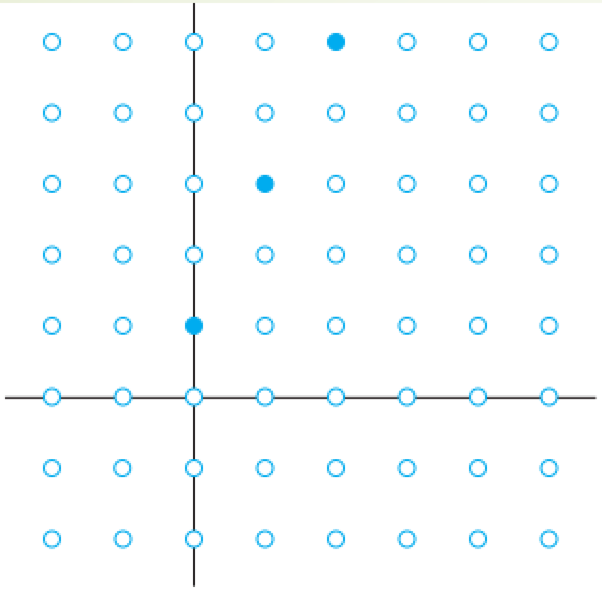

FIGURE 8 The Graph of $f(n) = 2n + 1$ from Z to Z.

FIGURE 9 The Graph of $f(x) = x^2$ from Z to Z.

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Sequences

A sequence is a discrete structure used to represent an ordered list. For example, 1, 2, 3, 5, 8 is a sequence with five terms and 1, 3, 9, 27, 81 ,.... ,3n, ... is an infinite sequence

- If the domain of a function is restricted to a subset of the set of integers (usually either the set {0, 1, 2,...} or the set {1, 2, 3,...}) to a set S, the function is called a ***sequence***

  - The domain is specifically the set **N** or the set Z^+.
  - a_n denotes the image of *n* – called a *term of the sequence*
  - Notation for whole sequence: $\{a_n\}$
  - a_n is called the n^{th} term or general term.

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EXAMPLES

- Let $\{a_n\}$ be a sequence, where

  $a_n = 1/n$ and $n \in \mathbf{Z}^+$

- What are the *terms* of the sequence?

  $a_1 = 1$

  $a_2 = 1/2$

  $a_3 = 1/3$

  $a_4 = 1/4$

  … … …

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Geometric progression

A *geometric progression* is a sequence of the form

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where the *initial term* a and the *common ratio* r are real numbers.

- Can be written as $f(x)=a \cdot r^x$

Example

- The sequences $\{b_n\}$ with $b_n=(-1)^n$, $\{c_n\}$ with $c_n=2*5^n$, $\{d_n\}$ with $d_n=6 *(1/3)^n$ are geometric progression
  - b_n : 1, -1, 1, -1, 1, …
  - c_n: 2, 10, 50, 250, 1250, …
  - $d_{n:}$ 6, 2, 2/3, 2/9, 2/27, …

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Arithmetic progression

An *arithmetic progression* is a sequence of the form

$$a, a+d, a+2d, \ldots, a+nd, \ldots$$

where the *initial term* a and the *common difference* d are real numbers.

Remark: An arithmetic progression is a discrete analogue of the linear function $f(x) = dx + a$.

Example

Let $\{a_n\}$ be a sequence, where a_n = -1 + 4n

– What type of progression is this? (Arithmetic)
– What is the initial term? (-1)
– What is the common ratio/difference? (4)
– What are the terms of the sequence? (-1, 3, 7, 11, …)

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- Let $\{t_n\}$ be a sequence, where $t_n = 7 - 3n$

  - What type of progression is this? (Arithmetic)
  - What is the initial term? (7)
  - What is the common ratio/difference? (-3)
  - What are the terms of the sequence? (7, 4, 1, -2, …)

String

Sequences of the form $a_1, a_2, \ldots, a_n$ are often used in computer science These finite sequences of bits are also called **strings** The **length** of the string S is the number of terms The **empty string**, denoted by **λ**, is the string has no terms and has length zero.

- The string abcd is a string of length four.

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Example

Find formulae for the sequences with the following first five terms: (a) 1, 1/2, 1/4, 1/8, 1/16 (b) 1, 3, 5, 7, 9 (c) 1, –1, 1, –1, 1.

(a)– What type of progression is this? (geometric progression)

– What is the initial term? (1)

– What is the common ratio/difference? (1/2)

What is the formula? ($a_n = \frac{1}{2^n}$)

(b) We note that each term is obtained by adding 2 to the previous term.

What type of progression is this? (Arithmetic)

– What is the initial term? (1)

– What is the common ratio/difference? (2)

What is the formula? (a_n = 2n+1)

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(c) The terms alternate between 1 and -1. The sequence with a

$$a_n = (-1)^n, n = 0, 1, 2 \dots$$

What is the initial term? (1)

– What is the common ratio/difference? (r = -1)

What is the formula? ($a_n = (-1)^n$ n=0,1,2...)

How can we produce the terms of a sequence if the first 10 terms are 5, 11, 17, 23, 29, 35, 41, 47, 53, 59?

Solution: Note that each of the first 10 terms of this sequence after the first is obtained by adding 6 to the previous term. (We could see this by noticing that the difference between consecutive terms is 6.) Consequently, the *nth* term could be produced by starting with 5 and adding 6 a total of $n - 1$ times; that is, a reasonable guess is that the *nth* term is $5 + 6(n - 1) = 6n - 1$. (This is an arithmetic progression with $a = 5$ and $d = 6$.)

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TABLE 1 Some Useful Sequences.

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ...
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, ...
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ...
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, ...
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, ...
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

Summations

A **summation** denotes the sum of the terms of a sequence.

from the sequence $\{a_n\}$. We use the notation

$$\sum_{j=m}^{n} a_j, \qquad \textstyle\sum_{j=m}^{n} a_j, \qquad \text{or} \qquad \textstyle\sum_{m \le j \le n} a_j$$

(read as the sum from $j = m$ to $j = n$ of a_j) to represent

$$a_m + a_{m+1} + \cdots + a_n.$$

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Upper limit

Index of Summation

Lower limit

$$\sum_{j=1}^{5} j^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$$

$$= 1 + 4 + 9 + 16 + 25$$

$$= 55$$

Example

- Express the sum of the first 100 terms of the sequence $\{a_n\}$ where $a_n = 1/n$, $n = 1, 2, 3, \ldots$

The lower limit for the index of summation is 1, and the upper limit is 100. We write this sum as

$$\sum_{j=1}^{100} \frac{1}{j}.$$

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- What is the value of $\sum_{k=4}^{8}(-1)^k$?

$$\sum_{k=4}^{8}(-1)^k = (-1)^4 + (-1)^5 + (-1)^6 + (-1)^7 + (-1)^8$$
$$= 1 + (-1) + 1 + (-1) + 1$$
$$= 1.$$

- What is the value of $\sum_{k=1}^{5} k^2$

$$\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1+4+9+16+25=55$$

- When shifting an index of summation, it is important to make the appropriate changes in the corresponding summand.

but want the index of summation to run between 0 and 4 rather than from 1 to 5. To do this, we let *k = j - 1*. Then the new summation index runs from 0 (because k = 1 - 0 = 0 when j = 1) to 4 (because k = 5 - 1 = 4 when j = 5), and the term j^2 becomes $(k + 1)^2$. Hence,

$$\sum_{j=1}^{5} j^2 = \sum_{k=0}^{4} (k+1)^2.$$

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Geometric Series

The sum of a geometric progression is called a geometric series

• Commonly used

$$S = \sum_{j=0}^{n} ar^j = ar^0 + ar^1 + ar^2 + \ldots + ar^n$$

$$\begin{aligned}
rS_n &= r\sum_{j=0}^{n} ar^j && \text{substituting summation formula for } S \\
&= \sum_{j=0}^{n} ar^{j+1} && \text{by the distributive property} \\
&= \sum_{k=1}^{n+1} ar^k && \text{shifting the index of summation, with } k = j+1 \\
&= \left(\sum_{k=0}^{n} ar^k\right) + (ar^{n+1} - a) && \text{removing } k = n+1 \text{ term and adding } k = 0 \text{ term} \\
&= S_n + (ar^{n+1} - a) && \text{substituting } S \text{ for summation formula}
\end{aligned}$$

Solving for S_n shows that if $r \neq 1$, then

$$S_n = \frac{ar^{n+1} - a}{r - 1}.$$

If $r = 1$, then the $S_n = \sum_{j=0}^{n} ar^j = \sum_{j=0}^{n} a = (n+1)a$.

$$\sum_{j=0}^{n} ar^j = \begin{cases} \dfrac{ar^{n+1} - a}{r - 1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$

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Double Summation

- Often used in programs

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}(i+2i+3i)$$

$$= \sum_{i=1}^{4} 6i = 6+12+18+24 = 60$$

$$\sum_{i=1}^{3}\sum_{j=1}^{2}(i-j)$$

$$\sum_{j=1}^{2}(i-j) = (i-1)+(i-2) = 2i-3$$

$$\sum_{i=1}^{3}(2i-3) = (2\cdot 1-3)+(2\cdot 2-3)+(2\cdot 3-3)$$

$$= -1+1+3 = 3$$

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TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty}, kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

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Examples

1. Find $\sum_{k=50}^{100} k^2$

$$\sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2 = \frac{100\cdot 101\cdot 201}{6} - \frac{49\cdot 50\cdot 99}{6} = 338350 - 40425 = 297925$$

$\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$ from Table 2

2. Let x be a real number with $|x|<1$, Find $\sum_{n=0}^{\infty} x^n$

$$\sum_{j=0}^{n} ar^j = \begin{cases} \frac{ar^{n+1}-a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}, \quad \sum_{n=0}^{k} x^n = \frac{x^{k+1}-1}{x-1}, \quad \sum_{n=0}^{\infty} x^n = \lim_{k\to\infty} \frac{x^{k+1}-1}{x-1} = \frac{-1}{x-1} = \frac{1}{1-x}$$

Differentiating both sides of $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$

$$\sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

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What are the values of these sums?

a) $\sum_{k=1}^{5} (k+1)$

b) $\sum_{j=0}^{4} (-2)^j$

c) $\sum_{i=1}^{10} 3$

d) $\sum_{j=0}^{8} (2^{j+1} - 2^j)$

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Thank
You!