Lecture 2

CH.1 The Foundations: Logic and Proofs

- Propositional Logic
- Predicate Logic
- Proofs

Ch.1 The Foundations: Logic and Proofs

• Propositional Logic

- Propositional Logic
- Applications of Propositional Logic
- Propositional Equivalences

1.1 Propositional Logic

Propositions : is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.

Examples

All the following declarative sentences are propositions.

1. Washington, D.C., is the capital of the United States of America. **(T)**
2. Toronto is the capital of Canada. **(F)**
3. 1 + 1 = 2. **(T)**
4. 2 + 2 = 3. **(F)**

The following sentences that are not propositions

1. What time is it?
2. Read this carefully.
3. $x + 1 = 2$.
4. $x + y = z$.

The truth (T) or falsity (F) of a proposition is called truth value

We use letters to denote **propositional variables**

Lecture 1

Logical Connectives and Truth Tables

simple propositions can be combined to form more complicated propositions called **compound propositions**.

The devices which are used to link pairs of propositions are called **logical connectives**

we first look at an operation which can be performed on a **single** proposition. This operation is called **negation**

If p symbolizes a proposition $\bar{p}$ (or $\sim p$ or $-p$ or $\neg p$) symbolizes the **negation** of p

$\neg p$ "It is not the case that p."

TABLE 1 The Truth Table for the Negation of a Proposition.

p	$\neg p$
T	F
F	T

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logical connectives

1. The *conjunction*

The conjunction of p and q, denoted by p ∧ q, is the proposition "p and q." **$p \wedge q$**

the Conjunction of Two Propositions.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

2. The disjunction

The *disjunction* of *p* and *q*, denoted by *p* ∨ *q*, is the proposition "*p* or *q*." **$p \vee q$**

the Disjunction of Two Propositions.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

- an **inclusive or**. A disjunction is true when **at least one** of the two propositions is true.
- an **exclusive or**. A disjunction is true when **exactly one** of the two propositions is true.

  denoted by $p \oplus q$

the Exclusive Or of Two Propositions.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Lecture 1

3. Conditional Statements

The *conditional statement* $p \rightarrow q$ is the proposition "if *p*, then *q*."

The statement $p \rightarrow q$ is called a conditional statement because $p \rightarrow q$ asserts that *q* is true on the condition that *p* holds.

the Conditional Statement $p \rightarrow q$.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- A conditional statement is also called an **implication**.
- "p is sufficient for q"
- "a necessary condition for p is q"
- "q unless ¬p"
- *p* only if *q*" says that *p* cannot be true when *q* is not true.

"If I am elected, then I will lower taxes."

When p is false, q may be either true or false, because the statement says nothing about the truth value of q

Lecture 1

Example

Let p be the statement "Maria learns discrete mathematics" and q the statement "Maria will find a good job." Express the statement $p \rightarrow q$ as a statement in English

Solution

$p \rightarrow q$ represents the statement

- "**If** Maria learns discrete mathematics, **then** she will find a good job."

There are many other ways to express this conditional statement in English

- "Maria will find a good job **when** she learns discrete mathematics."
- "For Maria to get a good job, **it is sufficient** for her to learn discrete mathematics."
- "Maria will find a good job **unless** she does **not** learn discrete mathematics."

Lecture 1

CONVERSE, CONTRAPOSITIVE, AND INVERSE

We can form some new conditional statements starting with a conditional statement $\mathbf{p \rightarrow q}$

- the **converse** of $p \rightarrow q$ is the proposition $q \rightarrow p$
- The **contrapositive** of $p \rightarrow q$ is the proposition $\neg q \rightarrow \neg p$
- **The inverse** of $p \rightarrow q$ is the proposition $\neg p \rightarrow \neg q$

The following truth table gives values of the conditional together with those for its converse, inverse and contrapositive.

p	q	$p \rightarrow q$	$q \rightarrow p$	$\bar{p} \rightarrow \bar{q}$	$\bar{q} \rightarrow \bar{p}$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Lecture 1

Example State the converse, inverse and contrapositive of the proposition 'If Jack plays his guitar then Sara will sing'.

Solution

We define: p : Jack plays his guitar
q : Sara will sing

so that : $p \rightarrow q$: If Jack plays his guitar then Sara will sing.

Converse: $q \rightarrow p$: If Sara will sing then Jack plays his guitar.
Inverse: $\bar{p} \rightarrow \bar{q}$: If Jack doesn't play his guitar then Sara won't sing.
Contrapositive: $\bar{q} \rightarrow \bar{p}$: If Sara won't sing then Jack doesn't play his guitar.

Quiz (1)

What are the contrapositive, the converse, and the inverse of the conditional statement? :

"The home team wins whenever it is raining?"

3. Biconditional Statements

The biconditional statement $p \leftrightarrow q$ is the proposition "p if and only if q."

- "p is necessary and sufficient for q"
- "if p then q, and conversely"
- "p iff q."

Biconditional $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Example

Let p be the statement "You can take the flight," and let q be the statement "You buy a ticket." Then $p \leftrightarrow q$ is the statement

"You can take the flight **if and only if** you buy a ticket."

Example

Consider the following propositions:

p : Mathematicians are generous.

q : Spiders hate algebra.

Write the compound propositions symbolized by:

(i) $p \vee \neg q$

(ii) $\neg(q \wedge p)$

(iii) $\neg p \rightarrow q$

(iv) $\neg p \leftrightarrow \neg q$.

Solution

(i) Mathematicians are generous or spiders don't hate algebra (or both).

(ii) It is not the case that spiders hate algebra and mathematicians are generous.

(iii) If mathematicians are not generous then spiders hate algebra.

(iv) Mathematicians are not generous if and only if spiders don't hate algebra.

Truth Tables of Compound Propositions

We can use these connectives to build up complicated compound propositions involving any number of propositional variables. We can use truth tables to determine the truth values of these compound propositions

Example

Construct the truth table of the compound proposition

$$(p \vee \neg q) \rightarrow (p \wedge q)$$

Solution

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Precedence of Logical Operators

the precedence levels of the logical operators, ¬, ∧, ∨, →, and ↔.

- ¬p ∧ q namely, (¬p) ∧ q, not the negation of the conjunction of p and q, namely ¬(p ∧ q).
- p ∧ q ∨ r means (p ∧ q) ∨ r rather than p ∧ (q ∨ r)
- p ∨ q → r is the same as (p ∨ q) → r

Precedence of Logical Operators.

Operator	*Precedence*
¬	1
∧	2
∨	3
→	4
↔	5

Logic and Bit Operations

Lecture 1

A bit is a symbol with two possible values, namely, 0 (zero) and 1 (one).

Computer **bit operations** correspond to the logical connectives. By replacing true by a one and false by a zero in the truth tables for the operators ∧, ∨, and ⊕,

Truth Value	*Bit*
T	1
F	0

We will also use the notation OR, AND, and XOR for the operators ∨, ∧, and ⊕, as is done in various programming languages.

Applications of Propositional Logic

1. Translating English Sentences

Example

How can this English sentence be translated into a logical expression?

"**You can access the Internet from campus** only if **you are a computer science major** or **you are not a freshman**."

Solution

it is possible to represent the sentence by a single propositional variable

a : You can access the Internet from campus

c : you are a computer science major

f : you are a freshman

this sentence can be represented as

$$a \rightarrow (c \vee \neg f).$$

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2. **System Specifications**

Translating sentences in natural into logical expressions is an essential part of specifying both hardware and software systems.

Example

Express the specification "The automated reply cannot be sent **when** the file system is full" using logical connectives.

let p denote "The automated reply can be sent"

q denote "The file system is full."

our specification can be represented by the conditional statement

$$q \rightarrow \neg p$$

3. **Boolean Searches**

4. **Logic Puzzles**

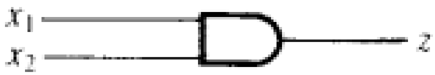
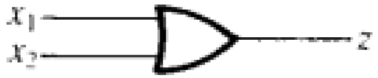
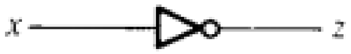
5. **Logic Circuits**

Lecture 1

5. **Logic Networks**

These are electronic devices which may be viewed as the basic functional components of a digital computer. **A logic gate**

three most important types of logic gates are the AND-gate, the OR-gate and the NOT-gate (or inverter).

	AND-gate	OR-gate	NOT-gate
Circuit symbol			
Input/output table	x_1 | x_2 | z 0 | 0 | 0 0 | 1 | 0 1 | 0 | 0 1 | 1 | 1	x_1 | x_2 | z 0 | 0 | 0 0 | 1 | 1 1 | 0 | 1 1 | 1 | 1	x | z 0 | 1 1 | 0
Boolean expression	$z = x_1 x_2$	$z = x_1 \oplus x_2$	$z = \bar{x}$

Lecture 1

Quiz (2)

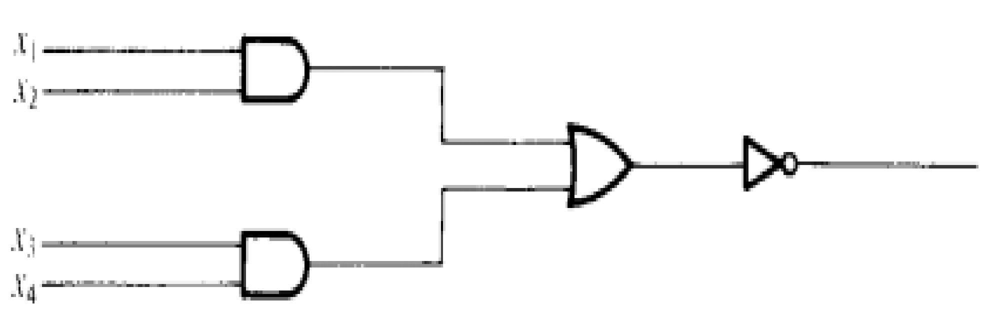
Determine whether each of the following is a tautology, a contradiction or neither:

1. $p \rightarrow (p \vee q)$
2. $(p \rightarrow q) \wedge (\neg p \vee q)$
3. $(p \vee q) \leftrightarrow (q \vee p)$
4. $(p \wedge q) \rightarrow p$
5. $(p \wedge q) \wedge (p \vee q)$
6. $(p \rightarrow q) \rightarrow (p \wedge q)$
7. $(\neg p \wedge q) \wedge (p \vee \neg q)$
8. $(p \rightarrow \neg q) \vee (\neg r \rightarrow p)$
9. $[p \rightarrow (q \wedge r)] \leftrightarrow [(p \rightarrow q) \wedge (p \rightarrow r)]$
10. $[(p \vee q) \rightarrow r] \oplus (\neg p \vee \neg q)$.

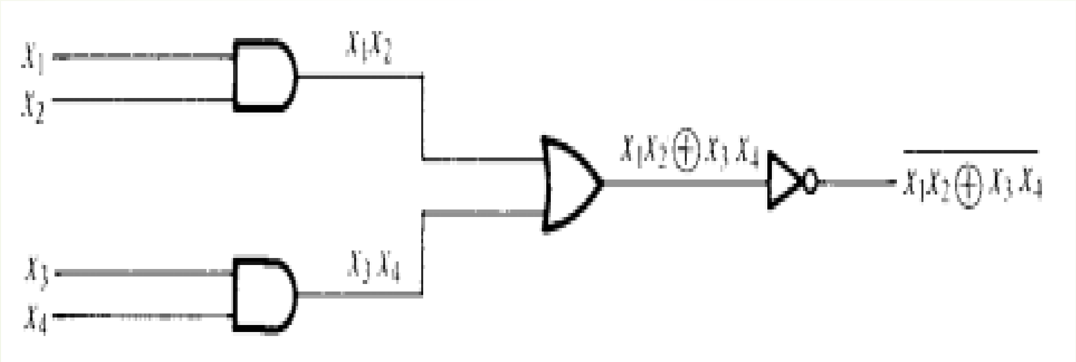
Lecture 1

Example

Give the Boolean expression for the output of the following system of gates.



Solution

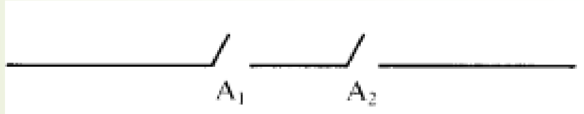


The final output is therefore $\overline{x_1x_2 \oplus x_3x_4}$.

Switching Circuits

A switch is an example of a two-state device, the two states being ‘on’ and ‘off’. A circuit which incorporates one or more switches is known as a switching circuit.

Consider now a circuit which contains two switches A1 and A2 connected as shown in the diagram below

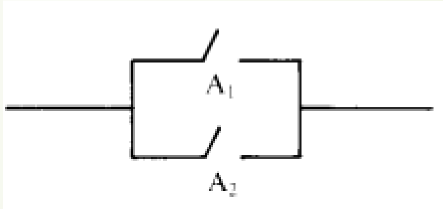


Switches connected to each other in this way are said to be in **series**.

x_1	x_2	$f(x_1, x_2)$
0	0	0
0	1	0
1	0	0
1	1	1

$f(x_1, x_2) = x_1x_2$

switching functions.



Two switches may alternatively be connected in **parallel**

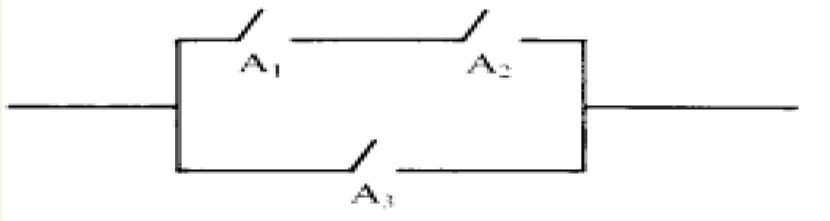
x_1	x_2	$g(x_1, x_2)$
0	0	0
0	1	1
1	0	1
1	1	1

$g(x_1, x_2) = x_1 \oplus x_2$

Lecture 1

Example

Define the switching function f for the circuit incorporating the following arrangement of switches.

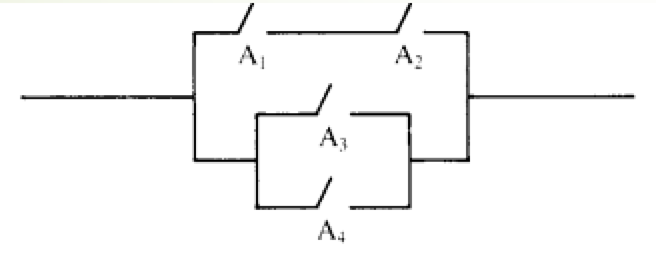


Solution

$$f(x_1, x_2, x_3) = f_1(x_1, x_2) \oplus f_2(x_3)$$
$$= x_1 x_2 \oplus x_3.$$

Example

Define the switching function f for the circuit incorporating the following arrangement of switches.



Solution

$$f(x_1, x_2, x_3, x_4) = f_1(x_1, x_2) \oplus f_2(x_3, x_4)$$
$$= x_1 x_2 \oplus x_3 \oplus x_4.$$

Lecture 1

Propositional Equivalences

We begin our discussion with a classification of compound propositions according to their possible truth values.

Tautology

A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it

p	$\bar{p}$	$p \vee \bar{p}$
T	F	T
F	T	T

contradiction

A compound proposition that is always false

p	q	$\bar{q}$	$p \wedge \bar{q}$	$\bar{p}$	$\bar{p} \vee q$	$(p \wedge \bar{q}) \wedge (\bar{p} \vee q)$
T	T	F	F	F	T	F
T	F	T	T	F	F	F
F	T	F	F	T	T	F
F	F	T	F	T	T	F

Logical Equivalences

Compound propositions that have the same truth values in all possible cases are called **logically equivalent**.

The compound propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

Example

Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

Morgan's Laws.

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

Solution

The truth tables for these compound propositions are

TABLE 3 Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Example

Show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent. This is the *distributive law* of disjunction over conjunction.

Solution

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

≡

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Idempotent laws

$$p \wedge p \equiv p$$
$$p \vee p \equiv p.$$

Commutative laws

$$p \wedge q \equiv q \wedge p$$
$$p \vee q \equiv q \vee p$$
$$p \oplus q \equiv q \oplus p$$
$$p \leftrightarrow q \equiv q \leftrightarrow p.$$

Associative laws

$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$
$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$
$$(p \oplus q) \oplus r \equiv p \oplus (q \oplus r)$$
$$(p \leftrightarrow q) \leftrightarrow r \equiv p \leftrightarrow (q \leftrightarrow r).$$

Absorption laws

$$p \wedge (p \vee q) \equiv p$$
$$p \vee (p \wedge q) \equiv p$$

Lecture 1

Distributive laws

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$
$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r).$$

Involution law

$$\neg(\neg p) \equiv p.$$

De Morgan's laws

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$
$$\neg(p \wedge q) \equiv \neg p \vee \neg q.$$

Identity laws

$$p \vee f \equiv p$$
$$p \wedge t \equiv p$$
$$p \vee t \equiv t$$
$$p \wedge f \equiv f.$$

Complement laws

$$p \vee \neg p \equiv T$$
$$p \wedge \neg p \equiv F$$
$$\neg F \equiv T$$
$$\neg T \equiv F.$$

Lecture 1

Some useful equivalences for compound propositions involving conditional statements and biconditional statements in Tables 7 and 8, respectively. The reader is asked to verify the equivalences in Tables 6–8 in the exercises

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Lecture 1

Replacement Rule

Suppose that we have two logically equivalent propositions $P1$ and $P2$, so that $P1 \equiv P2$. Suppose also that we have a compound proposition Q in which $P1$ appears. **The replacement rule** says that we may replace $P1$ by $P2$ and the resulting proposition is logically equivalent to Q.

Example

Show that $\neg(p \lor (\neg p \land q))$ and $\neg p \land \neg q$ are logically equivalent by developing a series of logical equivalences.

Solution

$\neg(p \lor (\neg p \land q))$	$\equiv \neg p \land \neg(\neg p \land q)$	by the second De Morgan law
	$\equiv \neg p \land [\neg(\neg p) \lor \neg q]$	by the first De Morgan law
	$\equiv \neg p \land (p \lor \neg q)$	by the double negation law
	$\equiv (\neg p \land p) \lor (\neg p \land \neg q)$	by the second distributive law
	$\equiv F \lor (\neg p \land \neg q)$	because $\neg p \land p \equiv F$
	$\equiv (\neg p \land \neg q) \lor F$	by the commutative law for disjunction
	$\equiv \neg p \land \neg q$	by the identity law for F

Consequently $\neg(p \lor (\neg p \land q))$ and $\neg p \land \neg q$ are logically equivalent.

Lecture 1

Example

Prove that $(\neg p \wedge q) \vee \neg (p \vee q) \equiv \neg p$

$(\neg p \wedge q) \vee \neg (p \vee q) \equiv (\neg p \wedge q) \vee (\neg p \wedge \neg q)$ (De Morgan's laws)

$\equiv \neg p \wedge (q \vee \neg q)$ (distributive laws)

$\equiv \neg p \wedge T$ (complement laws)

$\equiv \neg p.$ (identity laws)

Quiz (3)

Prove each of the following logical equivalences

1) $p \wedge [(p \vee q) \vee (p \vee r)] \equiv p.$
2) $q \wedge [(p \vee q) \wedge \neg (\neg q \wedge \neg p)] \equiv q.$

Propositional Satisfiability

A compound proposition is **satisfiable** if there is an assignment of truth values to its variables that makes it true.

when the compound proposition is false for all assignments of truth values to its variables, the compound proposition is **unsatisfiable**.

When we find a particular assignment of truth values that makes a compound proposition true, we have shown that it is satisfiable; such an assignment is called a solution of this particular satisfiability problem. However, to show that a compound proposition is unsatisfiable, we need to show that every assignment of truth values to its variables makes it false.

Lecture 1

Example

Determine the satisfiability of the following compound propositions:

1. $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$

Note that $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$ is true when the three variable p, q, and r have the same truth value

Hence, it is satisfiable as there is at least one assignment of truth values for p, q, and r is true and at least one is false that makes it true

2. $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$

Similarly, it is satisfiable as there is at least one assignment of truth values for p, q, and r that makes it true.

3. $[(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)] \wedge [(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)]$

For the first to be true, the three variables must have the same truth values, and for the second to be true, at least one of three variables must be true and at least one must be false. However, these conditions are contradictory. From these observations we conclude that no assignment of truth values makes it true . Hence, it is **unsatisfiable**.

Lecture 1